
Cuaderno de notas de trabajo

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Cuaderno 26

64 — Ecuaciones diferenciales de las
líneas de universo de las partículas
exploradoras del campo central
de Birkhoff en coordenadas
polares esféricas, con la longitud
de arco de Minkowski como
variable independiente.

70 — Ecuaciones diferenciales de las
trayectorias de las partículas exploradoras
del campo central de Birkhoff en
coordenadas polares esféricas con
el tiempo como variable independiente.

El Lagrangiano

$$L = e^{2MB^{-1}Q} [2BD - B^{-1}b^{-2}D^{-1}]$$

vale para el caso de un ~~punto~~ campo gravitacional generado por un punto masa, de masa M , en movimiento uniforme con un vector velocidad constante $\vec{V}$.

En el Lagrangiano L los símbolos significan lo siguiente:

$\vec{V}$ ——— vector velocidad constante del punto masa generador del campo;

$$B = \frac{1}{\sqrt{1-V^2}} ;$$

$\vec{r}$ ——— vector de posición de la partícula exploradora del campo con relación a la posición retardada del punto masa generador del mismo.

$$r = |\vec{r}|$$

$$R = \frac{1}{F - \vec{V} \cdot \vec{F}}$$

$\vec{v}$ — — — velocidad de la partícula exploradora del campo

$$b = \frac{1}{\sqrt{1 - v^2}}$$

$$D = 1 - (\vec{V} \cdot \vec{v})$$

$$\mathcal{L} = e^{2MB^{-1}R} [U(v^2, V^2, \vec{V} \cdot \vec{v})]$$

$$U[v^2, V^2, (\vec{V} \cdot \vec{v})] = [2BD - B^{-1}b^{-2}D^{-1}]$$

Programa:

Desarrollar en serie a U
en términos de potencias de V^2 , v^2 y $\vec{V} \cdot \vec{v}$

$$B = \frac{1}{\sqrt{1-\vec{V}^2}} = 1 + \left(-\frac{1}{2}\right)(-\vec{V}^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1 \cdot 2} (-\vec{V}^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1 \cdot 2 \cdot 3} (-\vec{V}^2)^3 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{1 \cdot 2 \cdot 3 \cdot 4} (-\vec{V}^2)^4 + \dots$$

$$B = 1 + \frac{1}{2} \vec{V}^2 + \frac{1 \cdot 3}{2 \cdot 4} \vec{V}^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \vec{V}^6 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \vec{V}^8 + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \vec{V}^{10} + \dots$$

$$D = 1 + \vec{V} \cdot \vec{v}$$

$$2BD = \left[2 + 2 \cdot \frac{1}{2} \vec{V}^2 + 2 \cdot \frac{1 \cdot 3}{2 \cdot 4} \vec{V}^4 + 2 \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \vec{V}^6 + 2 \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \vec{V}^8 + 2 \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \vec{V}^{10} \right. \\ \left. + 2(\vec{V} \cdot \vec{v}) + 2 \frac{1}{2} \vec{V}^2 (\vec{V} \cdot \vec{v}) + 2 \cdot \frac{1 \cdot 3}{2 \cdot 4} \vec{V}^4 (\vec{V} \cdot \vec{v}) + 2 \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \vec{V}^6 (\vec{V} \cdot \vec{v}) \right. \\ \left. + 2 \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \vec{V}^8 (\vec{V} \cdot \vec{v}) + \dots \right]$$

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$$B^{-1} = \sqrt{1 - V^2} = \left(1 + \frac{1}{2}(-V^2) + \frac{1}{2}(-\frac{1}{2})(-V^2)^2 + \frac{1}{6}(-\frac{1}{2})(-\frac{3}{2})(-V^2)^3 + \frac{1}{24}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-V^2)^4 + \frac{1}{120}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})(-V^2)^5 + \dots \right)$$

$$B^{-1} = \left(1 - \frac{1}{2}V^2 - \frac{1}{24}V^4 - \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}V^6 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}V^8 - \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}V^{10} \right)$$

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$$D^{-1} = \frac{1}{1 - (\vec{V} \cdot \vec{v})} = 1 + (\vec{V} \cdot \vec{v}) + (\vec{V} \cdot \vec{v})^2 + (\vec{V} \cdot \vec{v})^3 + (\vec{V} \cdot \vec{v})^4 + (\vec{V} \cdot \vec{v})^5 + \dots$$

$$B^{-1}D^{-1} = 1 - \frac{1}{2}V^2 - \frac{1}{24}V^4 - \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}V^6 + \dots + (\vec{V} \cdot \vec{v}) + \frac{1}{2}V^2(\vec{V} \cdot \vec{v}) + \frac{1}{24}V^4(\vec{V} \cdot \vec{v})^2 + \frac{1}{2}V^2(\vec{V} \cdot \vec{v})^2 + \dots - \frac{1}{2}V^2(\vec{V} \cdot \vec{v})^3 + \dots$$

$$B = \frac{1}{\sqrt{1-V^2}} = (1-V^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}(-V^2) + \frac{1}{2}(-V^2)^2 + \frac{1}{6}(-V^2)^3 + \dots$$

$$B = 1 + \frac{1}{2}V^2 + \frac{1 \cdot 3}{2 \cdot 4}V^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}V^6 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}V^{10} + \dots$$

$$D = (1 + \vec{V} \cdot \vec{V})$$

$$2BD = \left[2 + 2 \cdot \frac{1}{2}V^2 + 2 \cdot \frac{1 \cdot 3}{2 \cdot 4}V^4 + 2 \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}V^6 + 2 \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}V^{10} + \dots \right] + 2(\vec{V} \cdot \vec{V}) + 2 \cdot \frac{1}{2}V^2 + 2 \cdot \frac{1 \cdot 3}{2 \cdot 4}V^4 + 2 \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}V^6 + 2 \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}V^8 + \dots$$

Can probably

$$B^{-1} = (1 - V^2)^{-1/2} = 1 + \frac{1}{2}(-V^2) + \frac{1}{2}(-\frac{1}{2})(-V^2)^2 + \frac{1}{6}(-\frac{1}{2})(-V^2)^3 + \frac{1}{24}(-\frac{1}{2})(-V^2)^4 + \dots$$

$$B^{-1} = 1 - \frac{1}{2}V^2 + \frac{1}{8}V^4 - \frac{1}{16}V^6 + \frac{1}{128}V^8 - \frac{1}{2048}V^{10} + \dots$$

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$$D^2 = \frac{1}{\sqrt{1-V^2}}$$

$$D^{-1} = \frac{1}{1 - \frac{1}{2}V^2} = 1 + (\frac{1}{2}V^2) + (\frac{1}{2}V^2)^2 + (\frac{1}{2}V^2)^3 + (\frac{1}{2}V^2)^4 + (\frac{1}{2}V^2)^5 + \dots$$

$$B^{-1}D^{-1} = 1 - \frac{1}{2}V^2 - \frac{1}{4} \cdot \frac{1}{2}V^4 - \frac{1}{6} \cdot \frac{1}{2}V^6 - \frac{1}{8} \cdot \frac{1}{2}V^8 - \frac{1}{10} \cdot \frac{1}{2}V^{10} + (\frac{1}{2}V^2) - \frac{1}{2}V^2(\frac{1}{2}V^2) - \frac{1}{4} \cdot \frac{1}{2}V^4(\frac{1}{2}V^2) - \frac{1}{6} \cdot \frac{1}{2}V^6(\frac{1}{2}V^2) - \frac{1}{8} \cdot \frac{1}{2}V^8(\frac{1}{2}V^2) - \frac{1}{10} \cdot \frac{1}{2}V^{10}(\frac{1}{2}V^2) + (\frac{1}{2}V^2)^2 - \frac{1}{2}V^2(\frac{1}{2}V^2)^2 - \frac{1}{4} \cdot \frac{1}{2}V^4(\frac{1}{2}V^2)^2 - \frac{1}{6} \cdot \frac{1}{2}V^6(\frac{1}{2}V^2)^2 - \frac{1}{8} \cdot \frac{1}{2}V^8(\frac{1}{2}V^2)^2 + (\frac{1}{2}V^2)^3 - \frac{1}{2}V^2(\frac{1}{2}V^2)^3 - \frac{1}{4} \cdot \frac{1}{2}V^4(\frac{1}{2}V^2)^3 - \frac{1}{6} \cdot \frac{1}{2}V^6(\frac{1}{2}V^2)^3 - \frac{1}{8} \cdot \frac{1}{2}V^8(\frac{1}{2}V^2)^3 + (\frac{1}{2}V^2)^4 - \frac{1}{2}V^2(\frac{1}{2}V^2)^4 - \frac{1}{4} \cdot \frac{1}{2}V^4(\frac{1}{2}V^2)^4 - \frac{1}{6} \cdot \frac{1}{2}V^6(\frac{1}{2}V^2)^4 - \frac{1}{8} \cdot \frac{1}{2}V^8(\frac{1}{2}V^2)^4 + (\frac{1}{2}V^2)^5 - \frac{1}{2}V^2(\frac{1}{2}V^2)^5 - \dots$$

$$b^{-2} = 1 - V^2$$

$$-b^{-2} = -1 + V^2$$

$$\begin{aligned}
& -\frac{1}{2}V^2 + \frac{1}{4}V^4 + \frac{1}{6}V^6 + \frac{1}{8}V^8 + \frac{1}{10}V^{10} + \dots \\
& -(\vec{V} \cdot \vec{v}) + \frac{1}{2}V^2(\vec{V} \cdot \vec{v}) + \frac{1}{4}V^4(\vec{V} \cdot \vec{v}) + \frac{1}{6}V^6(\vec{V} \cdot \vec{v}) + \frac{1}{8}V^8(\vec{V} \cdot \vec{v}) + \dots \\
& + v^2 - (\vec{V} \cdot \vec{v})^2 + \frac{1}{2}V^2v^2 - (\vec{V} \cdot \vec{v})^3 + \frac{1}{4}V^4v^2 - (\vec{V} \cdot \vec{v})^4 + \frac{1}{6}V^6v^2 - (\vec{V} \cdot \vec{v})^5 + \dots \\
& + (\vec{V} \cdot \vec{v})v^2 - \frac{1}{2}V^2(\vec{V} \cdot \vec{v})v^2 + \frac{1}{4}V^4(\vec{V} \cdot \vec{v})v^2 + \frac{1}{6}V^6(\vec{V} \cdot \vec{v})v^2 + \frac{1}{8}V^8(\vec{V} \cdot \vec{v})v^2 + \dots \\
& - \frac{1}{4}V^4(\vec{V} \cdot \vec{v})^2 + \frac{1}{6}V^6(\vec{V} \cdot \vec{v})^2 - \frac{1}{8}V^8(\vec{V} \cdot \vec{v})^2 + \dots \\
& + (\vec{V} \cdot \vec{v})^3 - \frac{1}{2}V^2(\vec{V} \cdot \vec{v})^3 + \frac{1}{4}V^4(\vec{V} \cdot \vec{v})^3 - \frac{1}{6}V^6(\vec{V} \cdot \vec{v})^3 + \dots \\
& + (\vec{V} \cdot \vec{v})^4 - \frac{1}{2}V^2(\vec{V} \cdot \vec{v})^4 + \frac{1}{4}V^4(\vec{V} \cdot \vec{v})^4 - \frac{1}{6}V^6(\vec{V} \cdot \vec{v})^4 + \dots \\
& + (\vec{V} \cdot \vec{v})^5 - \frac{1}{2}V^2(\vec{V} \cdot \vec{v})^5 + \frac{1}{4}V^4(\vec{V} \cdot \vec{v})^5 - \frac{1}{6}V^6(\vec{V} \cdot \vec{v})^5 + \dots \\
& + (\vec{V} \cdot \vec{v})^6 - \frac{1}{2}V^2(\vec{V} \cdot \vec{v})^6 + \frac{1}{4}V^4(\vec{V} \cdot \vec{v})^6 - \frac{1}{6}V^6(\vec{V} \cdot \vec{v})^6 + \dots \\
& + (\vec{V} \cdot \vec{v})^7 - \frac{1}{2}V^2(\vec{V} \cdot \vec{v})^7 + \frac{1}{4}V^4(\vec{V} \cdot \vec{v})^7 - \frac{1}{6}V^6(\vec{V} \cdot \vec{v})^7 + \dots \\
& + (\vec{V} \cdot \vec{v})^8 - \frac{1}{2}V^2(\vec{V} \cdot \vec{v})^8 + \frac{1}{4}V^4(\vec{V} \cdot \vec{v})^8 - \frac{1}{6}V^6(\vec{V} \cdot \vec{v})^8 + \dots \\
& + (\vec{V} \cdot \vec{v})^9 - \frac{1}{2}V^2(\vec{V} \cdot \vec{v})^9 + \frac{1}{4}V^4(\vec{V} \cdot \vec{v})^9 - \frac{1}{6}V^6(\vec{V} \cdot \vec{v})^9 + \dots \\
& + (\vec{V} \cdot \vec{v})^{10} - \frac{1}{2}V^2(\vec{V} \cdot \vec{v})^{10} + \frac{1}{4}V^4(\vec{V} \cdot \vec{v})^{10} - \frac{1}{6}V^6(\vec{V} \cdot \vec{v})^{10} + \dots
\end{aligned}$$

Escalares in. significativos!

$$M, R, (\vec{V} \cdot \vec{r}), V^2, v^2, (\vec{V} \cdot \vec{v})$$

Se puede hacer $M=1$.

$$M=1. R, V^2, v^2, (\vec{V} \cdot \vec{v})$$

[illegible]

$$B^{-1} - T = \beta$$

$$2\beta R = 2R \sim R \sim R^2$$

$$2B-R = 2R - V^2R - \frac{1}{4}V^4R - \frac{1}{8}V^6R - \frac{5}{64}V^8R - \frac{7}{128}V^{10}R$$

$$\frac{25-27}{2} = \frac{22-22}{2} =$$

$$\begin{aligned} & \frac{1}{2} \left(\frac{1}{2} R^2 - 4V^2 R^2 + V^4 R^2 + \frac{1}{2} V^6 R^2 - \frac{5}{16} V^8 R^2 - \frac{7}{32} V^{10} R^2 \right) \\ & + \frac{1}{4} + \frac{1}{16} + \frac{5}{32} + \frac{2}{32} \end{aligned}$$

$$\frac{(2B^{-1}R)^2}{2!} = 2R^2 - 2V^2R^2.$$

$$\frac{(2B^{-1}R)^3}{3!}$$

$$B^{-3} = (1 - V^2)^{3/2} = 1 + \frac{3}{2}(+V^2) + \frac{3}{2}(+\frac{1}{2})V^4 + \frac{3}{2}(\frac{1}{2})(-\frac{1}{2})(-V^2)^3 + \frac{3}{2}(\frac{1}{2})(-\frac{1}{2})(-\frac{1}{2})(-\frac{1}{2})V^4$$

$$+ \frac{3}{2}(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})V^4 + \frac{3}{2}(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})V^4$$

$$a_i = \left(\frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^i} \right) v^j v^k$$

$$a_i = \left\{ v^k \frac{\partial}{\partial x^k} h_{ij} \right\} v^j - \frac{\partial}{\partial x^i} \{ h_{jk} v^j v^k \}$$

$$\hat{a} = \frac{dh_{ij}}{ds} v^i$$

$$\hat{a} = \frac{dh}{ds} \hat{v} - \nabla \{ (h \hat{v}) \cdot \hat{v} \}$$

$$h_{ij} = \tau \chi_{ij}$$

$\tau = \text{traza}$

$$\oint_A^B (h \hat{v}) \cdot \hat{v} ds = 0$$

$$2 \frac{d}{ds} (h \hat{v}) - \nabla \{ (h \hat{v}) \cdot \hat{v} \} = 0$$

$$2 \frac{dh}{ds} \hat{v} + 2 h \hat{a} - \nabla \{ (h \hat{v}) \cdot \hat{v} \} = 0$$

~~$\frac{dh}{ds}$~~

T_{--} ~ tensor en el espacio
tiempo

$$\dot{A} = \frac{dA}{dw} \quad w - \text{parámetro}$$

$$A = (t, x, y, z)$$

$$(T^{\hat{A}}) \cdot \hat{A} ; (T^{\hat{A}}) \cdot (T^{\hat{A}})$$

$$\cancel{\alpha \frac{d}{dw} [(T^{\hat{A}})] +}$$

$$\int_A^B \left[\alpha (T^{\hat{A}}) \cdot \hat{A} + \beta \sqrt{(T^{\hat{A}}) \cdot (T^{\hat{A}})} \right] dw = 0$$

$$\cancel{\alpha \frac{d}{dw} [(T^{\hat{A}})] + \alpha d}$$

Sea w un parámetro arbitrario.

Sea T un tensor T_{ij} en el espacio tiempo

Sea $\hat{r} = (t, x, y, z)$

$$\hat{v} = \frac{d\hat{r}}{dw}$$

Considérense los escalares:
 $\{(T\hat{v}) \cdot \hat{v}\}; (T\hat{v}) \cdot (T\hat{v})$.

$$\oint_A^B \{(T\hat{v}) \cdot \hat{v}\} dw = 0$$

$$\frac{d}{dw} \{2(T\hat{v})\} - \nabla \{(T\hat{v}) \cdot \hat{v}\} = 0$$

$$2 \frac{dT}{dw} \hat{v} + 2T \frac{d\hat{v}}{dw} - \nabla \{(T\hat{v}) \cdot \hat{v}\} = 0$$

$$\oint \{(T\hat{v}) \cdot \hat{v}\} dw = 0$$

$$2 \frac{dT}{dw} \hat{v} + 2T \frac{d\hat{v}}{dw} - \nabla \{(T\hat{v}) \cdot \hat{v}\} = 0$$

$$\delta \int_A^B (T_{ij})^2 d\omega = 0$$

$$\frac{\partial}{\partial \omega} [(T_{ij})^2] = 2(T_{ij})$$

$$T_{ij} = T_{11}v^1 + T_{12}v^2 + \dots$$

$$T_{ij} = \begin{pmatrix} T_1^1 v^1 + T_2^1 v^2 + T_3^1 v^3 + T_4^1 v^4 \\ T_1^2 v^1 + T_2^2 v^2 + T_3^2 v^3 + T_4^2 v^4 \\ T_1^3 v^1 + T_2^3 v^2 + T_3^3 v^3 + T_4^3 v^4 \\ T_1^4 v^1 + T_2^4 v^2 + T_3^4 v^3 + T_4^4 v^4 \end{pmatrix}$$

$$T_{ij} = \begin{pmatrix} T_i^1 v^i \\ T_i^2 v^i \\ T_i^3 v^i \\ T_i^4 v^i \end{pmatrix}$$

$$(T_{ij})^j = T_i^j v^j$$

$$(T^{\hat{1}}_v)^j = T^j_i v^i$$

$$(T^{\hat{1}}_v)^2 = \Delta_{mn} T^m_i v^i T^n_j v^j$$

$$(T^{\hat{1}}_v)^2 = \left[\Delta_{mn} T^m_i T^n_j \right] v^i v^j$$

$$\frac{\partial (T^{\hat{1}}_v)^2}{\partial v^k} =$$

$$\frac{\partial (T^{\hat{1}}_v)^2}{\partial v^k} = \left(\left[\Delta_{mn} T^m_k T^n_j \right] v^j + \left[\Delta_{mn} T^m_i T^n_k \right] v^i \right)$$

$$\frac{\partial (T^{\hat{1}}_v)^2}{\partial v^k} = 2 \left[\Delta_{mn} T^m_k T^n_j \right] v^j$$

$$\frac{\partial (T^{\hat{1}}_v)^2}{\partial v^k} = 2 T_{nk} (T^n_j v^j)$$

$$\frac{\partial (T^{\hat{1}})^2}{\partial \hat{v}} = 2 T (T^{\hat{1}})$$

$$\delta \int (T^{\hat{1}})^2 d\omega = 0$$

$$2 \frac{d}{d\omega} [T (T^{\hat{1}})] - 2 (T^{\hat{1}}) \cdot \nabla (T^{\hat{1}})$$

$$(T^{\hat{1}})^i = T_j^i \omega_j$$

$$(T^{\hat{1}})_i = T_{ij} \omega_j$$

$$(T^{\hat{1}})^2 = T_p^i \omega^p T_{iq} \omega^q$$

$$(T^{\hat{1}})^2 = (T_p^i T_{iq}) \omega^p \omega^q$$

Análisis del oscalar

$$(T^A)^2$$

$$\Delta_{ij} T_p^i \vee p T_q^j \vee q = (T^A)^2$$

$$(T^A)^2 = [\Delta_{ij} T_p^i T_q^j] \vee p \vee q$$

$$\Delta_{ij} T_p^i T_q^j \text{ es simétrico.}$$

demostración

$$\Delta_{ij} T_q^i T_p^j = \Delta_{ij} T_p^j T_q^i =$$

$$\rightarrow \Delta_{ji} T_p^j T_q^i = \Delta_{ij} T_p^i T_q^j$$

$$\Delta_{ij} T_q^i T_p^j = \Delta_{ij} T_p^i T_q^j$$

$$\frac{\partial}{\partial v^k} (T^{\hat{1}})^2 = \Delta_{ij} T_k^i T_q^j v^q + \Delta_{ij} T_p^i T_k^j v^p$$

$$\frac{\partial}{\partial v^k} (T^{\hat{1}})^2 = \left(\left(\Delta_{ij} T_k^i T_q^j \right) v^q + \Delta_{ij} T_q^i T_k^j v^q \right)$$

$$\frac{\partial}{\partial v^k} (T^{\hat{1}})^2 = \left[\left(\Delta_{ij} T_k^i T_q^j \right) v^q + \left(\Delta_{ji} T_q^j T_k^i \right) v^q \right]$$

$$\frac{\partial}{\partial v^k} (T^{\hat{1}})^2 = 2 \left[\Delta_{ij} T_k^i T_q^j \right] v^q$$

$$\frac{\partial}{\partial v^k} (T^{\hat{1}})^2 = T$$

$$\frac{\partial}{\partial v^k} (T^{\hat{1}})^2 = 2T (T^{\hat{1}})$$

$$[T(T^\dagger)]^k = T_i^k T_j^i v^j$$

~~∇_i~~

~~$\frac{\partial}{\partial x^k} [T(T^\dagger)]^k$~~

$$(T^\dagger)^2 = [\Delta_{ij} T_p^i T_q^j] v^p v^q$$

$$\frac{\partial}{\partial x^l} [(T^\dagger)^2] = \left[\left(\Delta_{ij} \frac{\partial T_p^i}{\partial x^l} T_q^j \right) v^p v^q + \Delta_{ij} T_p^i \frac{\partial T_q^j}{\partial x^l} v^p v^q \right]$$

$$\frac{\partial}{\partial x^l} (T^\dagger)^2 = \left[\left(\Delta_{ij} \frac{\partial T_p^i}{\partial x^l} T_q^j \right) v^p v^q + \Delta_{ji} T_q^j \frac{\partial T_p^i}{\partial x^l} v^q v^p \right]$$

$$\frac{\partial}{\partial x^l} (T^{\hat{a}})^2 = 2 \Delta_{ij} \frac{\partial T^i}{\partial x^l} T^j \quad \nu^p \nu^q \quad 19$$

$$\frac{\partial}{\partial x^l} (T^{\hat{a}})^2 = 2(T^{\hat{a}}) \left(\frac{\partial T^{\hat{a}}}{\partial x^l} \right)$$

$$\nabla (T^{\hat{a}})^2 = 2(T^{\hat{a}}) \cdot \nabla (T^{\hat{a}})$$

$$\nabla (T^{\hat{a}})^2 = 2 \{ (T^{\hat{a}}) \cdot \nabla \} (T^{\hat{a}})$$

$$(T^{\hat{a}})^2$$

$$\frac{\partial}{\partial \hat{a}} (T^{\hat{a}})^2 = 2 T (T^{\hat{a}})$$

$$\nabla (T^{\hat{a}})^2 = 2 \{ (T^{\hat{a}}) \cdot \nabla \} (T^{\hat{a}})$$

$$\begin{aligned} \frac{d}{d\omega} \left[\frac{\partial}{\partial \hat{a}} (T^{\hat{a}})^2 \right] &= 2 \frac{dT}{d\omega} (T^{\hat{a}}) + 2 T \left(\frac{dT^{\hat{a}}}{d\omega} \right) \\ &\quad + 2 T \left(T \frac{d\hat{a}}{d\omega} \right) \end{aligned}$$

$$\sum_{\alpha} \frac{M_{\alpha} [2V_i^{\alpha} V_j^{\alpha} - \Delta_{ij}]}{(\hat{V}^{\alpha} \cdot \hat{F}^{\alpha})}$$

Campo Central.

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Principio Variacional para
la variable independiente s .

$$\delta \int_A^B e^{\frac{2M}{r}} (1 + v^2) dt = 0$$

$$\delta \int_A^B e^{\frac{2M}{r}} \left(\frac{dt^2 + dx^2 + dy^2 + dz^2}{dt} \right) = 0$$

$$\delta \int_A^B e^{\frac{2M}{r}} \left(\frac{t'^2 + x'^2 + y'^2 + z'^2}{t''} \right) ds = 0$$

$$\delta \int_A^B e^{\frac{2M}{r}} \frac{\left(\frac{dt}{dw} \right)^2 + \left(\frac{dx}{dw} \right)^2 + \left(\frac{dy}{dw} \right)^2 + \left(\frac{dz}{dw} \right)^2}{\left(\frac{dt}{dw} \right)} dw = 0$$

$$L = e^{\frac{2M}{r}} \frac{\left(\frac{dt}{dw}\right)^2 + \left(\frac{dx}{dw}\right)^2 + \left(\frac{dy}{dw}\right)^2 + \left(\frac{dz}{dw}\right)^2}{\left(\frac{dt}{dw}\right)}$$

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$$\frac{\partial L}{\partial \left(\frac{dx}{dw}\right)} = 2e^{\frac{2M}{r}} \frac{\left(\frac{dx}{dw}\right)}{\left(\frac{dt}{dw}\right)}$$

$$\frac{\partial L}{\partial x} = -2e^{\frac{2M}{r}} \frac{x}{r^3}$$

$$\frac{d}{dw} \left[e^{\frac{2M}{r}} \frac{\left(\frac{dx}{dw}\right)}{\left(\frac{dt}{dw}\right)} \right] + \cancel{2}e^{\frac{2M}{r}} \frac{x}{r^3} = 0$$

$$\cancel{e^{\frac{2M}{r}}} \frac{dt}{dw} \frac{d^2x}{dw^2} - \frac{dx}{dw} \frac{dt}{dw^2} + \cancel{e^{\frac{2M}{r}}} \left(-2 \frac{Mx}{r^3} \right) = 0$$

$$+ \cancel{e^{\frac{2M}{r}}} \frac{\left(\frac{dx}{dw}\right)}{\left(\frac{dt}{dw}\right)} \left[-\frac{2M}{r^2} \frac{dr}{dw} \right] + \cancel{e^{\frac{2M}{r}}} \frac{x}{r^3}$$

$$\frac{d^2 x}{dw^2} \frac{dt}{dw} - \frac{dx}{dw} \frac{d^2 t}{dw^2} - \frac{2M}{r^2} \frac{dx}{dw} \frac{dr}{dw} \frac{dt}{dw}$$

$$+ \frac{x}{r^3} \left(\frac{dt}{dw} \right)^2 = 0$$

$$\frac{d^2 x}{dw^2} \frac{dt}{dw} - \frac{dx}{dw} \frac{d^2 t}{dw^2} = \frac{2M}{r^2} \frac{dx}{dw} \frac{dr}{dw} \frac{dt}{dw} - \frac{x}{r^3} \left(\frac{dt}{dw} \right)^2$$

$$\frac{\partial \mathcal{L}}{\partial t} \quad \frac{\partial \mathcal{L}}{\partial t}$$

$$\frac{\partial \mathcal{L}}{\partial \left(\frac{dt}{dw} \right)} = e^{\frac{2M}{r}} \left[1 - \frac{\left(\frac{dx}{dw} \right)^2 + \left(\frac{dy}{dw} \right)^2 + \left(\frac{dz}{dw} \right)^2}{\left(\frac{dt}{dw} \right)^2} \right]$$

$$\frac{\partial \mathcal{L}}{\partial t} = 0$$

$$\frac{d}{dw} \left[e^{\frac{2M}{r}} \left[1 - \frac{\left(\frac{dx}{dw} \right)^2 + \left(\frac{dy}{dw} \right)^2 + \left(\frac{dz}{dw} \right)^2}{\left(\frac{dt}{dw} \right)^2} \right] \right] = 0$$

$$x' \quad x'' = -\frac{Mx}{r^3} - \frac{2Mx}{r^3}(x'^2 + y'^2 + z'^2) + \frac{Mx'r'}{r^2}$$

$$y' \quad y'' = -\frac{My}{r^3} - \frac{2My}{r^3}(x'^2 + y'^2 + z'^2) + \frac{My'r'}{r^2}$$

$$z' \quad z'' = -\frac{Mz}{r^3} - \frac{2Mz}{r^3}(x'^2 + y'^2 + z'^2) + \frac{Mz'r'}{r^2}$$

$$t'^2 - x'^2 - y'^2 - z'^2 = 1$$

$$t't'' - x'x'' - y'y'' - z'z'' = 0$$

$$t't'' = \left[-\frac{M(x'x + y'y + z'z)}{r^3} - \frac{2M(x'x' + y'y' + z'z')}{r^3}(x'^2 + y'^2 + z'^2) + \frac{Mr'}{r^2}(x'^2 + y'^2 + z'^2) \right]$$

$$t't'' = \left[-\frac{Mr'}{r^2} - \frac{2Mr'}{r^2}(x'^2 + y'^2 + z'^2) + \frac{Mr'}{r^2}(x'^2 + y'^2 + z'^2) \right]$$

$$t't'' = -\frac{Mr'}{r^2} [1 + x'^2 + y'^2 + z'^2]$$

$$t't'' = -\frac{Mr'}{r^2} t'^2$$

$$t'' = - \frac{M r' t'}{r^2}$$

$$\frac{t''}{t'} = - \frac{M r'}{r^2}$$

$$\ln t' = \frac{M}{r} + \ln A$$

$$t' = A e^{\frac{M}{r}}$$

$$t' t'' = - \frac{M r' t'^2}{r^2}$$

$$t'' = - \frac{M r' t'}{r^2}$$

$$x' x'' = - \frac{M x x'}{r^3} - \frac{2 M x x'}{r^3} (x'^2 + y'^2 + z'^2) + \frac{M x'^2 r'}{r^2}$$

$$y' y'' = - \frac{M y y'}{r^3} - \frac{2 M y y'}{r^3} (x'^2 + y'^2 + z'^2) + \frac{M y'^2 r'}{r^2}$$

$$z' z'' = - \frac{M z z'}{r^3} - \frac{2 M z z'}{r^3} (x'^2 + y'^2 + z'^2) + \frac{M z'^2 r'}{r^2}$$

$$x' x'' + y' y'' + z' z'' =$$

$$- \frac{M r r'}{r^3} - \frac{2 M r r'}{r^3} (x'^2 + y'^2 + z'^2) + M r'$$

$$+ \frac{M r'}{r^2} (x'^2 + y'^2 + z'^2)$$

$$x'x'' + y'y'' + z'z'' = -\frac{Mr'}{r^2} - \frac{Mr'}{r^2}(x'^2 + y'^2 + z'^2)$$

$$x'x'' + y'y'' + z'z'' = -\frac{Mr'}{r^2}(1 + x'^2 + y'^2 + z'^2)$$

$$x'x'' + y'y'' + z'z'' = -\frac{Mr't'^2}{r^2}$$

$$t'^2 - x'^2 - y'^2 - z'^2 = 1$$

$$t't'' - x'x'' - y'y'' - z'z'' = 0$$

$$x'x'' + y'y'' + z'z'' = t't''$$

$$\delta \int_A^B e^{\frac{2M}{r}} (1 + v^2) dt$$

Variable independiente r .

$$v^2 = \frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2}{dt^2}$$

$$v^2 = \frac{1 + r^2 \theta'^2 + r^2 \sin^2 \theta \varphi'^2}{t'^2}$$

$$t' = \frac{d}{dr}$$

$$\delta \int e^{\frac{2M}{r}} \frac{1 + r^2 \theta'^2 + r^2 \sin^2 \theta \varphi'^2}{t'^2} t' dr = 0$$

$$\frac{\partial \mathcal{L}}{\partial t'} = e^{\frac{2M}{r}} \left[1 - \frac{r^2 \theta'^2 + r^2 \sin^2 \theta \varphi'^2}{r^2} \right]$$

$$\delta \int e^{\frac{2M}{r}} [1 + v^2] dt$$

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$$x'' = -\frac{Mx}{r^3} - \frac{2Mx}{r^3} (x'^2 + y'^2 + z'^2) + \frac{Mx'r'}{r^2}$$

$$t'' = -\frac{Mt'r'}{r^2}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r' = \frac{xx' + yy' + zz'}{r}$$

$$r'' = \frac{r}{r^2} \left[x \right]$$

$$r'' = \left[-\frac{r'}{r^2} (xx' + yy' + zz') + \frac{1}{r} [xx'' + yy'' + zz'' + x'^2 + y'^2 + z'^2] \right]$$

$$r'' = \left[\frac{r' r r'}{r^2} + \frac{1}{r} [xx'' + yy'' + zz''] + \frac{1}{r} [x'^2 + y'^2 + z'^2] \right]$$

$$r'' = \left[\frac{r'^2}{r} + \frac{1}{r} [x'^2 + y'^2 + z'^2] + \frac{1}{r} [xx'' + yy'' + zz''] \right]$$

comp prob

~~$$z = r \cos \theta$$~~

~~$$\cos \theta = \frac{z}{r}$$~~

~~$$-\sin \theta \theta' = \frac{r z' - z r'}{r^2}$$~~

~~$$\cos \theta = \frac{z}{r}$$~~

$$z = r \cos \theta$$

$$z' = r' \cos \theta - r \sin \theta \theta'$$

$$z'' = \left[r'' \cos \theta - 2r' \sin \theta \theta' - r \cos \theta \theta'^2 - r \sin \theta \theta'' \right]$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r' = \frac{xx' + yy' + zz'}{r}$$

$$r'' = \left[-\frac{r'}{r^2} (xx' + yy' + zz') + \frac{x'^2 + y'^2 + z'^2}{r} + \frac{xx'' + yy'' + zz''}{r} \right]$$

$$r'' = \left[-\frac{r'}{r^2} r r' + \frac{x'^2 + y'^2 + z'^2}{r} + \frac{1}{r} (xx'' + yy'' + zz'') \right]$$

$$r'' = \left[-\frac{r'^2}{r} + \frac{1}{r} (x'^2 + y'^2 + z'^2) + \frac{1}{r} (xx'' + yy'' + zz'') \right]$$

Comprobando

$$x x'' + y y'' + z z'' =$$

$$- \frac{M r^2}{r^3} - \frac{2M r^2}{r^3} (x'^2 + y'^2 + z'^2)$$

$$+ \frac{M r'}{r^2} (r r')$$

$$x x'' + y y'' + z z'' = \left[-\frac{M}{r} - \frac{2M}{r} (x'^2 + y'^2 + z'^2) \right] + \frac{M r'}{r}$$

$$r'' = \left[-\frac{r'}{r} + \frac{1}{r} [x'^2 + y'^2 + z'^2] - \frac{M}{r^2} - \frac{2M}{r^2} (x'^2 + y'^2 + z'^2) \right] + \frac{M r'}{r^2}$$

$$r'' = \begin{bmatrix} -\frac{r'^2}{r} + \frac{1}{r}[t'^2 - 1] \\ -\frac{M}{r^2} - \frac{2M}{r^2}[t'^2 - 1] \\ + \frac{Mr'^2}{r^2} \end{bmatrix}$$

Comprabado
pág 5.6

$$\cos \theta = \frac{z}{r}$$

$$-\sin \theta \theta' = \frac{rz' - zr'}{r^2}$$

$$z = r \cos \theta$$

$$z' = r' \cos \theta - r \theta' \sin \theta$$

$$z'' = \begin{bmatrix} r'' \cos \theta - 2r' \theta' \sin \theta \\ -r \theta'' \sin \theta - r \theta'^2 \cos \theta \end{bmatrix}$$

$$\theta'' = \begin{bmatrix} \frac{-z''}{r \sin \theta} + \frac{r'' \cot \theta}{r} - \frac{2r' \theta'}{r} \\ -\theta'^2 \cot \theta \end{bmatrix}$$

$$z = r \cos \theta$$

$$z' = r' \cos \theta - r \sin \theta \theta'$$

$$z'' = \left[\begin{aligned} &r'' \cos \theta - r' \sin \theta \theta' \\ &- r' \sin \theta \theta' - r \cos \theta \theta'^2 \\ &- r \sin \theta \theta'' \end{aligned} \right]$$

$$r \sin \theta \theta'' = \left[\begin{aligned} &-z'' + r'' \cos \theta - 2r' \theta' \sin \theta \\ &- r \cos \theta \theta'^2 \end{aligned} \right] \quad \text{ ~~$- r \theta''$~~ }$$

$$\theta'' = \left[\begin{aligned} &-\frac{z''}{r \sin \theta} + \frac{r''}{r} \cot \theta - \frac{2r' \theta'}{r} \\ &-\cot \theta \theta'^2 \end{aligned} \right]$$

$$z'' = \left[\begin{aligned} &\frac{M \cos \theta}{r^2} - \frac{2M \cos \theta}{r^2} (t'^2 - 1) + \\ &+ \frac{M r'}{r^2} [r' \cos \theta - r \sin \theta \theta'] \end{aligned} \right]$$

$$z'' = \left[-\frac{M \cos \theta}{r^2} + \frac{2M \cos \theta}{r^2} t'^2 + \frac{M r'^2}{r^2} \cos \theta - \frac{M r' \theta'}{r} \sin \theta \right]$$

$$z'' = \left[+\frac{M \cos \theta}{r^2} - \frac{2M \cos \theta}{r^2} t'^2 + \frac{M r'^2}{r^2} \cos \theta - \frac{M r' \theta'}{r} \sin \theta \right]$$

$$\begin{aligned} \theta'' = & \left[-\frac{M}{r^3} \cot \theta + \frac{2M \cot \theta}{r^3} t'^2 \right. \\ & - \frac{M r'^2}{r^3} \cot \theta + \frac{M r' \theta'}{r^2} \\ & - \frac{r'^2}{r^2} \cot \theta + \frac{\cot \theta}{r^2} (t'^2 - 1) \\ & - \frac{M}{r^3} \cot \theta - \frac{2M}{r^3} \cot \theta (t'^2 - 1) \\ & + \frac{M r'^2}{r^3} \cot \theta - \frac{2 r' \theta'}{r} \\ & \left. - \cot \theta \theta'^2 \right] \end{aligned}$$

$$\theta'' = \cancel{\frac{2M}{r^3}} \cot \theta +$$

$$\theta'' = \left[\frac{Mr' \theta'}{r^2} - \frac{r'^2}{r^2} \cot \theta + \frac{\cot \theta}{r^2} [t'^2 - 1] \right. \\ \left. - \frac{2r' \theta'}{r} - \theta'^2 \cot \theta \right]$$

$$x = r \sin \theta \cos \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$\varphi = \tan \varphi$$

$$\frac{y}{x} = \tan \varphi$$

$$\varphi = \tan^{-1} \frac{y}{x}$$

$$\varphi' = \frac{\frac{xy' - yx'}{x^2}}{1 + \frac{y^2}{x^2}}$$

$$\varphi' = \frac{xy' - yx'}{x^2 + y^2}$$

$$\varphi'' = \left[\frac{-2(xx' + yy')(xy' - yx')}{(x^2 + y^2)^2} + \frac{xy'' - yx''}{x^2 + y^2} \right]$$

$$\varphi'' = \left[-\frac{2}{(x^2 + y^2)^2} (xx' + yy')(xy' - yx') + \frac{xy'' - yx''}{x^2 + y^2} \right] \checkmark$$

$$x^2 + y^2 = r^2 \sin^2 \theta \checkmark$$

$$y_{x'} = \left(\begin{aligned} & r r' \sin^2 \theta \sin \varphi \cos \varphi \\ & + r^2 \theta' \sin \theta \cos \theta \sin \varphi \cos \varphi \\ & - r^2 \varphi' \sin^2 \theta \sin^2 \varphi \end{aligned} \right)$$

$$x = r \sin \theta \cos \varphi$$

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$$x' = r' \sin \theta \cos \varphi + r \theta' \cos \theta \cos \varphi - r \varphi' \sin \theta \sin \varphi$$

$$xx' = \left[\begin{aligned} &r r' \sin^2 \theta \cos^2 \varphi \\ &+ r^2 \theta' \sin \theta \cos \theta \cos^2 \varphi \\ &- r^2 \varphi' \sin^2 \theta \sin \varphi \cos \varphi \end{aligned} \right]$$

$$y = r \sin \theta \sin \varphi$$

$$y' = r' \sin \theta \sin \varphi + r \theta' \cos \theta \sin \varphi + r \varphi' \sin \theta \cos \varphi$$

$$yy' = \left[\begin{aligned} &r r' \sin^2 \theta \sin^2 \varphi \\ &+ r^2 \theta' \sin \theta \cos \theta \sin^2 \varphi \\ &+ r^2 \varphi' \sin^2 \theta \sin \varphi \cos \varphi \end{aligned} \right]$$

$$xx' + yy' = r r' \sin^2 \theta + r^2 \theta' \sin \theta \cos \theta$$

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$$xy' = \left[\begin{aligned} &r r' \sin^2 \theta \sin \varphi \cos \varphi \\ &+ r^2 \theta' \sin \theta \cos \theta \sin \varphi \cos \varphi \\ &+ r^2 \varphi' \sin^2 \theta \cos^2 \varphi \end{aligned} \right]$$

$$-\frac{2}{(x^2+y^2)^2} (xx' + yy') (xy' - yx') =$$

$$-2 \frac{r'\varphi'}{r} - 2\varphi'\theta' \cot \theta$$

$$xy' - yx' = r^2 \varphi' \sin^2 \theta$$

comprova-se
pág. 62

$$-\frac{2}{(x^2+y^2)^2} (xx' + yy') (xy' - yx') =$$

$$-\frac{2}{r^4 \sin^4 \theta} [rr' \sin^2 \theta + r^2 \theta' \sin \theta \cos \theta] r^2 \varphi' \sin^2 \theta$$

$$-\frac{2}{(x^2+y^2)^2} (xx' + yy') (xy' - yx') =$$

$$-\frac{2r r' \varphi' \sin^2 \theta + 2r^2 \theta' \varphi' \sin \theta \cos \theta}{r^4 \sin^4 \theta}$$

$$-\frac{2r r' \varphi' \sin^2 \theta + 2r^2 \theta' \varphi' \sin \theta \cos \theta}{r^4 \sin^4 \theta}$$

$$-\frac{2}{(x^2+y^2)^2} (xx' + yy') (xy' - yx') =$$

$$-\frac{2r r' \varphi' \sin^2 \theta + 2r^2 \theta' \varphi' \sin \theta \cos \theta}{r^4 \sin^4 \theta}$$

Falta: $\frac{xy'' - yx''}{x^2 + y^2}$

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$$xy'' = -\frac{Mxy}{r^3} - \frac{2Mxy}{r^3}(x'^2 + y'^2 + z'^2) + \frac{Mxy'r'}{r^2}$$

$$yx'' = -\frac{Mxy}{r^3} - \frac{2Mxy}{r^3}(x'^2 + y'^2 + z'^2) + \frac{Mx'y'r'}{r^2}$$

$$xy'' - yx'' = \frac{Mr'}{r^2}(xy' - x'y)$$

$$\ln(xy' - x'y) = \ln A - \frac{M}{r}$$

$$xy' - x'y = Ae^{-\frac{M}{r}}$$

$$\frac{xy'' - yx''}{x^2 + y^2} = \frac{Mr'}{r^4 \sin^2 \theta} (xy' - x'y)$$

$$xy' - x'y = r^2 \varphi' \sin^2 \theta$$

$$\frac{xy'' - yx''}{x^2 + y^2} = \frac{Mr^2 r' \varphi' \sin^2 \theta}{r^4 \sin^2 \theta}$$

$$\frac{xy'' - yx''}{x^2 + y^2} = \frac{Mr' \varphi'}{r^2}$$

Comprobando
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$$q'' = \frac{Mr'\varphi'}{r^2} - 2\frac{r'\varphi'}{r} - 2\varphi'\theta' \cot \theta$$

Completed
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$$g' = \left[-2rr' \dot{\theta} \sin^2 \theta - 2r^2 \dot{\theta} \dot{\phi} \sin \theta \cos \theta \right. \\ \left. + \frac{Mr' \dot{\phi}}{r^2} \right]$$

Las ecuaciones diferenciales de las líneas de universo de las partículas exploradoras del campo del central de Birkhoff en coordenadas polares esféricas.

$$t'' = - \frac{Mt'r'}{r^2} ;$$

$$r'' = \left[- \frac{r'^2}{r} + \frac{1}{r} (t'^2 - 1) \right. \\ \left. - \frac{M}{r^2} - \frac{2M}{r^2} (t'^2 - 1) \right. \\ \left. + \frac{Mr'^2}{r^2} \right]$$

Comp robado
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#

$$\theta'' = \left[\frac{Mr'\theta'}{r^2} - \frac{r'^2}{r^2} \cot\theta + \frac{\cot\theta}{r^2} (t'^2 - 1) \right. \\ \left. - \frac{2r'\theta'}{r} - \theta'^2 \cot\theta \right]$$

Comprovado pág. 59.

$$\varphi'' = \left[-2rr'\varphi' \sin^2\theta - 2r\theta'\varphi' \sin\theta \cos\theta \right. \\ \left. + \frac{Mr'\varphi'}{r^2} \right]$$

Ver pág. 41' o 63

$$\frac{t''}{t'} = -\frac{Mr'}{r^2}$$

$$\ln t' = \frac{CM}{r}$$

$$t' = e^{\frac{CM}{r}}$$

$$\oint_A^B h_{ij} v^i v^j v^k ds = 0$$

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Eliminar la variable s de las ecuaciones que expresan $t'', r'', \theta'', \varphi''$ y calcular $\ddot{r}, \ddot{\theta}, \ddot{\varphi}$.

$$\dot{r} = \frac{r'}{t'}$$

$$\bullet = \frac{1}{t'} \frac{d}{ds}$$

$$\ddot{r} = \frac{1}{t'} \frac{t' r'' - r' t''}{t'^2}$$

$$\ddot{r} = \frac{r''}{t'^2} - \dot{r} \frac{t''}{t'^2}$$

$$\ddot{\theta} = \frac{\theta''}{t'^2} - \dot{\theta} \frac{t''}{t'^2}$$

$$\ddot{\varphi} = \frac{\varphi''}{t'^2} - \dot{\varphi} \frac{t''}{t'^2}$$

$$\frac{r''}{t'^2} = \left[-\frac{\dot{r}^2}{r} + \frac{1}{r} - \frac{1}{rt'^2} - \frac{M}{r^2 t'^2} - \frac{2M}{r^2} + \frac{2M}{r^2 t'^2} + \frac{M \dot{r}^2}{r^2} \right]$$

$$\frac{t''}{t'^2} = \cancel{\frac{2M}{r^2}} - \frac{M \dot{r}^2}{r^2}$$

$$\ddot{r} = \left[-\frac{\dot{r}^2}{r} + \frac{1}{r} - \frac{1}{rt'^2} - \frac{M}{r^2 t'^2} - \frac{2M}{r^2} + \frac{2M}{r^2 t'^2} + \frac{2M \dot{r}^2}{r^2} \right]$$

$$\text{Ez} \quad ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

~~$$ds^2 = t'^2 - x'^2 - y'^2 - z'^2$$~~

$$ds^2 = dt^2 [1 - v^2]$$

$$t'^2 = \frac{1}{1 - v^2}$$

$$\ddot{r} = \left[-\frac{\dot{r}^2}{r} + \frac{1}{r} - \frac{1}{r}(1-v^2) - \frac{M}{r^2}(1-v^2) \right. \\ \left. - \frac{2M}{r^2} + \frac{2M}{r^2}(1-v^2) + \frac{2M\dot{r}^2}{r^2} \right]$$

$$\ddot{r} = -\frac{\dot{r}^2}{r^2} + \frac{v^2}{r} - \frac{M}{r^2} - \frac{M}{r^2}v^2 + \frac{2M\dot{r}^2}{r^2}$$

$$\boxed{\ddot{r} = -\frac{\dot{r}^2}{r} + \frac{v^2}{r} - \frac{M}{r^2}(1+v^2) + \frac{2M\dot{r}^2}{r^2}}$$

Comprovado página 66.

$$\frac{\ddot{\theta}}{t^2} = \left[\frac{M\dot{r}\dot{\theta}}{r^2} - \frac{\dot{r}^2}{r^2}\cot\theta + \frac{\cot\theta}{r^2} - \frac{\cot\theta}{r^2 t^{1/2}} \right. \\ \left. - \frac{2\dot{r}\dot{\theta}}{r} - \dot{\theta}^2 \cot\theta \right]$$

$$\dot{\theta} \frac{t''}{t^2} = -\frac{M\dot{r}\dot{\theta}}{r^2}$$

$$\ddot{\theta} = \left[\frac{2M\dot{r}\dot{\theta}}{r^2} - \frac{\dot{r}^2}{r^2}\cot\theta + \frac{\cot\theta}{r^2} - \frac{\cot\theta}{r^2(1-v^2)} \right. \\ \left. - \frac{2\dot{r}\dot{\theta}}{r} - \dot{\theta}^2 \cot\theta \right]$$

$$\ddot{\theta} = \left[\frac{2M\dot{r}\dot{\theta}}{r^2} - \frac{\dot{r}^2}{r^2} \cot \theta + \frac{\cot \theta}{r^2} v^2 \right. \\ \left. - \frac{2\dot{r}\dot{\theta}}{r} - \dot{\theta}^2 \cot \theta \right]$$

Comprobado pág. 67

~~$$\frac{\dot{\varphi}''}{t^2} = -2r\dot{r}\dot{\varphi}\sin^2\theta - 2r^2\dot{\theta}\dot{\varphi}\sin\theta\cos\theta$$~~

~~$$+ \frac{M\dot{r}\dot{\varphi}}{r^2}$$~~

~~$$\dot{\varphi} \frac{t''}{t^2} = - \frac{M\dot{r}\dot{\varphi}}{r^2}$$~~

~~$$\ddot{\varphi} = \frac{2M\dot{r}\dot{\varphi}}{r^2} - 2r\dot{r}\dot{\varphi}\sin^2\theta - 2r^2\dot{\theta}\dot{\varphi}\sin\theta\cos\theta$$~~

$$\frac{\varphi''}{t^2} = \frac{M\dot{r}\dot{\varphi}}{r^2} - \frac{2\dot{r}\dot{\varphi}}{r} - 2\dot{\varphi}\dot{\theta}\cot\theta$$

$$\dot{\varphi} \frac{t''}{t^2} = - \frac{M\dot{r}\dot{\varphi}}{r^2}$$

$$\ddot{\varphi} = \frac{2M\dot{r}\dot{\varphi}}{r^2} - \frac{2\dot{r}\dot{\varphi}}{r} - 2\dot{\varphi}\dot{\theta}\cot\theta$$

Campo Central
de Birkhoff.

Coordenadas
Polares

Esfericas

Variable independiente
del tiempo.

$$\ddot{r} = -\frac{\dot{r}^2}{r} + \frac{v^2}{r} - \frac{M}{r^2}(1+v^2) + \frac{2M\dot{r}^2}{r^2}$$

Comprobado pág. 50.

$$\ddot{\theta} = \left[-\frac{\dot{r}^2}{r^2} \cot \theta + \frac{\cot \theta}{r^2} v^2 - \frac{2\dot{r}\dot{\theta}}{r} \right. \\ \left. - \dot{\theta}^2 \cot \theta + \frac{2M\dot{r}\dot{\theta}}{r^2} \right]$$

comprobado pág. 67

$$\ddot{\varphi} = -\frac{2\dot{r}\dot{\varphi}}{r} - 2\dot{\varphi}\dot{\theta}\cot\theta + \frac{2M\dot{r}\dot{\varphi}}{r^2}$$

Comprobado pág. 53

$$\ddot{\varphi} = -\frac{2\dot{r}\dot{\varphi}}{r} - 2\dot{\varphi}\dot{\theta}\cot\theta + \frac{2M\dot{r}\dot{\varphi}}{r^2}$$

Comprobado pág. 68

↑

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$$\int e^{\frac{2M}{r}} (1 + \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) dt = 0$$

$$\oint_A^B e^{\frac{2M}{r}} [1 + \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2] dt = 0$$

$$\frac{\partial L}{\partial \dot{r}} = 2e^{\frac{2M}{r}} \dot{r}$$

$$\frac{\partial L}{\partial r} = \left[-\frac{2M}{r^2} e^{\frac{2M}{r}} (1 + v^2) + e^{\frac{2M}{r}} (2r \dot{\theta}^2 + 2r \sin^2 \theta \dot{\phi}^2) \right]$$

$$\frac{d}{dt} \left(2e^{\frac{2M}{r}} \dot{r} \right) = 2e^{\frac{2M}{r}} \ddot{r} + 4 \left(-\frac{M \dot{r}}{r^2} \right) \dot{r} e^{\frac{2M}{r}}$$

$$2e^{\frac{2M}{r}} \ddot{r} - \frac{4M \dot{r}^2}{r^2} e^{\frac{2M}{r}} + \frac{2M}{r^2} e^{\frac{2M}{r}} (1 + v^2) = 0$$

$$\ddot{r} = + \frac{2M \dot{r}^2}{r^2} - \frac{M}{r^2} (1 + v^2)$$

$$Q = \left[\cancel{2e^{\frac{2M}{r}} \ddot{r}} - \cancel{\frac{4M\dot{r}^2}{r^2} e^{\frac{2M}{r}}} + \cancel{\frac{2M}{r^2} e^{\frac{2M}{r}}} (1+v^2) \right. \\ \left. \rightarrow \cancel{e^{\frac{2M}{r}}} (2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\varphi}^2) \right]$$

$$\ddot{r} = \left[\frac{2M\dot{r}^2}{r^2} - \frac{M}{r^2} (1+v^2) \right. \\ \left. + r\dot{\theta}^2 + r\sin^2\theta\dot{\varphi}^2 \right]$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2$$

$$\frac{v^2}{r} = \frac{\dot{r}^2}{r} + r\dot{\theta}^2 + r\sin^2\theta\dot{\varphi}^2$$

$$\ddot{r} = \left[\frac{2M\dot{r}^2}{r^2} - \frac{M}{r^2} (1+v^2) \right. \\ \left. + \frac{v^2}{r} - \frac{\dot{r}^2}{r} \right]$$

Comprobado pág. 48

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 2r^2 e^{\frac{2M}{r}} \dot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = 2r^2 e^{\frac{2M}{r}} \sin \theta \cos \theta \dot{\varphi}^2$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \left[2r\dot{r} e^{\frac{2M}{r}} \dot{\theta} + 2r^2 \left(-\frac{2M\dot{r}}{r^2} \right) e^{\frac{2M}{r}} \dot{\theta} + 2r^2 e^{\frac{2M}{r}} \ddot{\theta} \right]$$

$$2r^2 \ddot{\theta} = \left[-2r\dot{r} \dot{\theta} + 4M\dot{r} \dot{\theta} + 2r^2 \sin \theta \cos \theta \dot{\varphi}^2 \right]$$

$$\ddot{\theta} = \left[-\frac{\dot{r} \dot{\theta}}{r} + \frac{2M\dot{r} \dot{\theta}}{r^2} + \sin \theta \cos \theta \dot{\varphi}^2 \right]$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = 2e^{\frac{2M}{r}} r^2 \sin^2 \theta \dot{\varphi} \quad \parallel \quad \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \quad 52$$

$$\frac{d}{dt} \left[2e^{\frac{2M}{r}} r^2 \sin^2 \theta \dot{\varphi} \right] = \cancel{\text{something}} \quad \text{O}$$

$$\boxed{e^{\frac{2M}{r}} r^2 \sin^2 \theta \dot{\varphi} = C_1}$$

$$\cancel{\frac{2M}{r^2}}$$

$$\left[\begin{aligned} & -\frac{2M \dot{r}}{r^2} e^{\frac{2M}{r}} r^2 \sin^2 \theta \dot{\varphi} + 2e^{\frac{2M}{r}} r \dot{r} \sin^2 \theta \dot{\varphi} \\ & + 2e^{\frac{2M}{r}} r^2 \sin \theta \cos \theta \dot{\varphi} \dot{\theta} \\ & + e^{\frac{2M}{r}} r^2 \sin^2 \theta \ddot{\varphi} \end{aligned} \right] = 0$$

$$\begin{aligned} r^2 \sin^2 \theta \ddot{\varphi} &= \cancel{r^2 \sin^2 \theta} \cdot 2M \sin^2 \theta \dot{\varphi} \\ & - 2r \dot{r} \sin^2 \theta \dot{\varphi} \\ & + 2r^2 \sin \theta \cos \theta \dot{\varphi} \dot{\theta} \\ & \quad \cancel{r^2 \sin^2 \theta \dot{\varphi}} \end{aligned}$$

$$\ddot{\varphi} = \left[\cancel{\frac{2M}{r^2}} + \frac{2M\dot{r}\dot{\varphi}}{r^2} - 2\frac{\dot{r}\dot{\varphi}}{r} - 2\dot{\varphi}\dot{\theta}\cot\theta \right]$$

$$\ddot{\varphi} = \frac{2M\dot{r}\dot{\varphi}}{r^2} - \frac{2\dot{r}\dot{\varphi}}{r} - 2\dot{\varphi}\dot{\theta}\cot\theta$$

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~~Cambio de variable independiente~~

Las ecuaciones diferenciales
de las líneas de universo de
las partículas exploradoras
en el campo central de
Birkhoff en coordenadas
polares esféricas.

Ecuaciones de partida.

$$x'' = -\frac{Mx}{r^3} - \frac{2Mx}{r^3}(t'^2 - 1) + \frac{Mx'r'}{r^2};$$

$$y'' = -\frac{My}{r^3} - \frac{2My}{r^3}(t'^2 - 1) + \frac{My'r'}{r^2};$$

$$z'' = -\frac{Mz}{r^3} - \frac{2Mz}{r^3}(t'^2 - 1) + \frac{Mz'r'}{r^2};$$

$$t'' = -\frac{Mt'r'}{r^2}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r' = \frac{xx' + yy' + zz'}{r}$$

$$r'' = \left[-\frac{r'}{r^2} (xx' + yy' + zz') + \frac{x'^2 + y'^2 + z'^2}{r} + \frac{xx'' + yy'' + zz''}{r} \right]$$

$$r'' = \left[-\frac{r'}{r^2} r r' + \frac{(t'^2 - 1)}{r} + \frac{1}{r} [xx'' + yy'' + zz''] \right]$$

$$r'' = \left[-\frac{r'^2}{r} + \frac{(t'^2 - 1)}{r} + \frac{1}{r} [xx'' + yy'' + zz''] \right]$$

$$xx'' + yy'' + zz'' = -\frac{M}{r} - \frac{2M}{r}(t'^2 - 1) + \frac{Mr' r r'}{r^2}$$

$$xx'' + yy'' + zz'' = -\frac{M}{r} - \frac{2M}{r}(t'^2 - 1) + \frac{Mr'^2}{r}$$

$$r'' = \left[-\frac{r'^2}{r} + \frac{(t'^2 - 1)}{r} - \frac{M}{r^2} - \frac{2M}{r^2}(t'^2 - 1) + \frac{Mr'^2}{r^2} \right]$$

comprovação pág. 34

$$z = r \cos \theta$$

$$\cos \theta = \frac{z}{r}$$

$$-\sin \theta \theta' = \frac{rz' - zr'}{r^2}$$

$$-\sin \theta \theta' = \frac{z'}{r} - \frac{zr'}{r^2}$$

$$\sin \theta \theta' = -\frac{z'}{r} + \frac{zr'}{r^2}$$

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$$\sin \theta \theta'' + \cos \theta \theta'^2 = \left[-\frac{z''}{r} + \frac{z'r'}{r^2} + \frac{z'r'}{r^2} + \frac{zr''}{r^2} - \frac{2zr'^2}{r^3} \right]$$

$$\sin \theta \theta'' = \left[-\cos \theta \theta'^2 - \frac{z''}{r} + \frac{2z'r'}{r^2} + \frac{zr''}{r^2} - \frac{2zr'^2}{r^3} \right]$$

$$z'' = -\frac{M \cos \theta}{r^2} - \frac{2M \cos \theta}{r^2} (t'^2 - 1) + \frac{M z'r'}{r^2}$$

$$z = r \cos \theta$$

$$z' = r' \cos \theta - r \sin \theta \theta'$$

$$z'r' = r'^2 \cos \theta - r r' \theta' \sin \theta$$

$$\frac{M z'r'}{r^2} = \frac{M r'^2}{r^2} \cos \theta - \frac{M r' \theta'}{r} \sin \theta$$

$$z'' = \left[-\frac{M \cos \theta}{r^2} - \frac{2M \cos \theta}{r^2} (t'^2 - 1) + \frac{M r'^2}{r^2} \cos \theta - \frac{M r' \theta'}{r} \sin \theta \right]$$

$$\begin{aligned} \sin \theta \theta'' = & \left[\cos \theta \theta'^2 - \frac{r'^2}{r^2} \cos \theta \right. \\ & - \frac{2r'\theta'}{r} \sin \theta \\ & + \frac{(t'^2 - 1)}{r^2} \cos \theta \\ & \cancel{\frac{M}{r^3} (t'^2 - 1) \cos \theta} \\ & \left. + \frac{Mr'\theta'}{r^2} \sin \theta \right] \end{aligned}$$

$$\begin{aligned} \theta'' = & \left[\cot \theta \theta'^2 - \frac{r'^2}{r^2} \cot \theta \right. \\ & - \frac{2r'\theta'}{r} + \frac{(t'^2 - 1)}{r^2} \cot \theta \\ & \cancel{\frac{M}{r^3} (t'^2 - 1) \cot \theta} \\ & \left. + \frac{Mr'\theta'}{r^2} \right] \end{aligned}$$

Comprobado pag 37.

$$x = r \sin \theta \cos \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$\tan \varphi = \frac{y}{x}$$

$$\varphi = \tan^{-1} \frac{y}{x}$$

$$\varphi' = \frac{\frac{xy' - yx'}{x^2}}{1 + \frac{y^2}{x^2}}$$

$$\boxed{\varphi' = \frac{xy' - yx'}{x^2 + y^2}}$$

$$\varphi'' = \frac{xy'' - yx''}{x^2 + y^2} - 2 \frac{(xy' - yx')}{(x^2 + y^2)^2} (xx' + yy')$$

$$\boxed{\varphi'' = \frac{xy'' - yx''}{x^2 + y^2} - 2 \frac{(xy' - yx')(xx' + yy')}{(x^2 + y^2)^2}}$$

$$xy'' - yx'' =$$

$$-\frac{M_{xy}}{r^3} - \frac{2M_{xy}}{r^3}(t'^2 - 1) + \frac{M_{xy}'r'}{r^2} + \frac{M_{xy}}{r^3} + \frac{M_{xy}}{r^3}(t'^2 - 1) - \frac{M_{x'y}r'}{r^2}$$

$$xy'' - yx'' = \frac{Mr'}{r^2}(xy' - x'y)$$

$$x = r \sin \theta \cos \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$x' = r' \sin \theta \cos \varphi + r \theta' \cos \theta \cos \varphi - r \varphi' \sin \theta \sin \varphi$$

$$y' = r' \sin \theta \sin \varphi + r \theta' \cos \theta \sin \varphi + r \varphi' \sin \theta \cos \varphi$$

$$xy' = r r' \sin^2 \theta \sin \varphi \cos \varphi + r^2 \theta' \sin \theta \cos \theta \sin \varphi \cos \varphi + r^2 \varphi' \sin^2 \theta \cos^2 \varphi$$

$$- x'y = - r r' \sin^2 \theta \sin \varphi \cos \varphi - r^2 \theta' \sin \theta \cos \theta \sin \varphi \cos \varphi + r^2 \varphi' \sin^2 \theta \sin^2 \varphi$$

$$xy' - x'y = r^2 \varphi' \sin^2 \theta$$

$$x^2 = r^2 \sin^2 \theta \cos^2 \varphi$$

$$y^2 = r^2 \sin^2 \theta \sin^2 \varphi$$

$$x^2 + y^2 = r^2 \sin^2 \theta$$

$$xy'' - yx'' = \frac{Mr'}{r^2} r^2 \varphi' \sin^2 \theta$$

$$xy'' - yx'' = Mr' \varphi' \sin^2 \theta$$

$$\frac{xy'' - yx''}{x^2 + y^2} = \frac{Mr' \varphi'}{r^2}$$

Comprobado
pág 41

$$xy' - yx' = r^2 \varphi' \sin^2 \theta$$

Comprobado
pág. 40

$$xx' + yy' = rr' \sin^2 \theta + r^2 \theta' \sin \theta \cos \theta$$

Comprobado pág 39

$$(xy' - yx')(xx' + yy') = r^3 r' \varphi' \sin^4 \theta + r^4 \varphi' \theta' \sin^3 \theta \cos \theta$$

$$\frac{(xy' - yx')(xx' + yy')}{(x^2 + y^2)^2} = \frac{r' \varphi'}{r} + \varphi' \theta' \cot \theta$$

$$\varphi'' = \frac{Mr'\varphi'}{r^2} - \frac{2r'\varphi'}{r} - 2\varphi'\theta'\cot\theta$$

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comprova da pag 41'

Ecuaciones diferenciales de
la línea de universo de una
partícula exploradora del
campo central de Birkhoff
en coordenadas polares
esféricas con la variable independi-
s. (longitud de arco de Minkowski)

$$t'' = - \frac{M t' r'}{r^2} \quad \text{pág. 42}$$

$$r'' = \left[- \frac{r'^2}{r} + \frac{t'^2 - 1}{r} - \frac{M}{r^2} - \frac{2M}{r^2} (t'^2 - 1) + \frac{M r'^2}{r^2} \right]; \quad \text{pág. 34}$$

$$\theta'' = \left[- \theta'^2 \cot \theta - \frac{r'^2}{r^2} \cot \theta - \frac{2r'\theta'}{r} + \frac{(t'^2 - 1)}{r^2} \cot \theta + \frac{M r'\theta'}{r^2} \right]; \quad \text{pág. 37}$$

$$\varphi'' = - \frac{2r'\varphi'}{r} - 2\varphi'\theta'\cot\theta + \frac{M r'\varphi'}{r^2}; \quad \text{pág. 41'}$$

comprobadas

~~$$t^2 - x'^2 - y'^2 - z'^2$$~~

$$cdt^2 - dx^2 - dy^2 - dz^2 = ds^2$$

$$cdt^2(1 - v^2) = ds^2$$

$$dt \sqrt{1 - v^2} = ds$$

$$t' = \frac{1}{\sqrt{1 - v^2}}$$

Cambio de la variable independiente de s a t .

$$\dot{r} = \frac{r'}{t'}$$

$$\dot{} = \frac{1}{t'} \frac{d}{ds}$$

$$\dot{} = \frac{1}{t'} \quad /$$

$$\ddot{r} = \frac{1}{t'} \frac{t' r'' - r' t''}{t'^2}$$

$$\ddot{r} = \frac{r''}{t'^2} - \frac{t''}{t'^2} \dot{r}$$

$$\ddot{r} = \frac{r''}{t'^2} - \dot{r} \frac{t''}{t'^2} \quad \text{pág. 44.}$$

$$\cancel{\ddot{r}} = -\frac{\dot{r}^2}{r} + \frac{t'^2 - 1}{r} - \frac{M}{r^2} - \frac{2M}{r^2} (t'^2 - 1)$$

$$\ddot{r} = \left[-\frac{\dot{r}^2}{r} + \frac{1}{r} - \frac{1}{t'^2 r} - \frac{M}{t'^2 r^2} \right. \\ \left. - \frac{2M}{r^2} + \frac{2M}{t'^2 r^2} + \frac{M \dot{r}^2}{r^2} + \frac{M \dot{r}^2}{r^2} \right]$$

$$\ddot{r} = \left[-\frac{\dot{r}^2}{r} + \frac{1}{r} - \frac{1}{r}(1-v^2) - \frac{M}{r^2}(1-v^2) \right. \\ \left. - \frac{2M}{r^2} + \frac{2M}{r^2}(1-v^2) + 2\frac{M\dot{r}^2}{r^2} \right]$$

$$\ddot{r} = -\frac{\dot{r}^2}{r} + \frac{v^2}{r} - \frac{M}{r^2} - \frac{Mv^2}{r^2} + 2M\frac{\dot{r}^2}{r^2}$$

$$\ddot{r} = -\frac{\dot{r}^2}{r} + \frac{v^2}{r} - \frac{M}{r^2}(1+v^2) + \frac{2M\dot{r}^2}{r^2}$$

Comprobo página 46.

$$\ddot{\theta} = \frac{\theta''}{t'^2} - \dot{\theta} \frac{t''}{t'^2}$$

$$\frac{\theta''}{t'^2} = \left[-\dot{\theta}^2 \cot \theta - \frac{\dot{r}^2}{r^2} \cot \theta - \frac{2\dot{r}\dot{\theta}}{r} \right. \\ \left. + \frac{\cot \theta}{r^2} - \frac{\cot \theta}{t'^2 r^2} + \frac{M\dot{r}\dot{\theta}}{r^2} \right]$$

$$\frac{\theta''}{t'^2} = -\dot{\theta}^2 \cot \theta - \frac{\dot{r}^2}{r^2} \cot \theta - \frac{2\dot{r}\dot{\theta}}{r} \\ + \frac{\cot \theta}{r^2} - \frac{\cot \theta}{r^2}(1-v^2) + \frac{M\dot{r}\dot{\theta}}{r^2}$$

$$\frac{t''}{t'^2} = - \frac{Mr}{r^2}$$

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$$-\ddot{\theta} \frac{t''}{t'^2} = + \frac{Mr\dot{\theta}}{r^2}$$

$$\ddot{\theta} = \left[-\dot{\theta}^2 \cot \theta - \frac{\dot{r}^2}{r^2} \cot \theta - \frac{2\dot{r}\dot{\theta}}{r} + \frac{\cot \theta}{r^2} - \frac{\cot \theta}{r^2} (1-v^2) + \frac{2Mr\dot{\theta}}{r^2} \right]$$

$$\ddot{\theta} = -\dot{\theta}^2 \cot \theta - \frac{\dot{r}^2}{r^2} \cot \theta - \frac{2\dot{r}\dot{\theta}}{r} + \frac{\cot \theta}{r^2} v^2 + \frac{2Mr\dot{\theta}}{r^2}$$

Comprobado pág. 47

$$\ddot{\varphi} = \frac{\varphi''}{t'^2} - \dot{\varphi} \frac{t''}{t'^2}$$

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$$\frac{\varphi''}{t'^2} = -\frac{2\dot{r}\dot{\varphi}}{r} - 2\dot{\varphi}\dot{\theta}\cot\theta + \frac{M\dot{r}\dot{\varphi}}{r^2}$$

$$-\dot{\varphi} \frac{t''}{t'^2} = + \frac{M\dot{r}\dot{\varphi}}{r^2}$$

$$\frac{\varphi''}{t'^2} - \dot{\varphi} \frac{t''}{t'^2} = -\frac{2\dot{r}\dot{\varphi}}{r} - 2\dot{\varphi}\dot{\theta}\cot\theta + \frac{2M\dot{r}\dot{\varphi}}{r^2}$$

$$\ddot{\varphi} = -\frac{2\dot{r}\dot{\varphi}}{r} - 2\dot{\varphi}\dot{\theta}\cot\theta + \frac{2M\dot{r}\dot{\varphi}}{r^2}$$

Comprobado pág. 48

$$\ddot{\theta} = \left[-\cancel{\ddot{\theta} \cot \theta} - \frac{\dot{r}^2}{r^2} \cot \theta - \frac{2\dot{r}\dot{\theta}}{r} + \left(\frac{\dot{r}^2}{r^2} \cot \theta + \cancel{\ddot{\theta} \cot \theta} + \sin \theta \cos \theta \dot{\varphi}^2 \right) + \frac{2M\dot{r}\dot{\theta}}{r^2} \right]$$

$$\cancel{\ddot{\theta}} = \cancel{r} + \frac{2\dot{r}\dot{\theta}}{r} + \sin^2 \theta \dot{\varphi}^2 + \frac{2M\dot{r}\dot{\theta}}{r^2}$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \sin \theta \cos \theta \dot{\varphi}^2 + \frac{2M\dot{r}\dot{\theta}}{r^2}$$

comprobado pág 81

Ecuaciones diferenciales de las trayectorias de las partículas exploradoras del campo central de Birkhoff en coordenadas polares esféricas con la variable independiente el tiempo.

$$\ddot{r} = -\frac{\dot{r}^2}{r} + \frac{v^2}{r} - \frac{M}{r^2}(1+v^2) + \frac{2M\dot{r}^2}{r^2}$$

Comprobado pág. 80

$$\ddot{\theta} = -\dot{\theta}^2 \cot \theta - \frac{\dot{r}^2}{r^2} \cot \theta - \frac{2\dot{r}\dot{\theta}}{r} + \frac{\cot \theta}{r^2} v^2 + \frac{2M\dot{r}\dot{\theta}}{r^2}$$

$$\ddot{\varphi} = -\frac{2\dot{r}\dot{\varphi}}{r} - 2\dot{\varphi}\dot{\theta} \cot \theta + \frac{2M\dot{r}\dot{\varphi}}{r^2}$$

Comprobado pág. 82

Elimínese el tiempo t de las dos primeras ecuaciones de la pág. 64. y ~~expresese r~~ y obténase una ecuación diferencial para r ~~en función de~~ con la variable independiente s .

$$t'' = - \frac{M t' r'}{r^2}$$

$$\frac{t''}{t'} = - \frac{M r'}{r^2}$$

$$\ln t' = + \frac{M}{r} + \ln A$$

$$t' = A e^{\frac{M}{r}}$$

$$r'' = \left[- \frac{r'^2}{r} + \frac{A^2 e^{\frac{2M}{r}} - 1}{r} - \frac{M}{r^2} \right. \\ \left. - \frac{2M}{r^2} \left[A^2 e^{\frac{2M}{r}} - 1 \right] + \frac{M r'^2}{r^2} \right]$$

$$r'' = \left[-\frac{r'^2}{r} - \frac{1}{r} - \frac{M}{r^2} + \frac{2M}{r^2} + \frac{M r'^2}{r^2} \right. \\ \left. + \frac{A^2 e^{\frac{2M}{r}}}{r} - \frac{2A^2 M e^{\frac{2M}{r}}}{r^2} \right]$$

$$r'' = F(r) r'^2 + G(r)$$

~~$$r'' = \left[-\frac{1}{r} + \frac{M}{r^2} \right] r'^2 \\ + \left[-\frac{1}{r} - \frac{M}{r^2} \right]$$~~

$$r'' = \left[-\frac{r'^2}{r} - \frac{1}{r} + \frac{M}{r^2} + \frac{M r'^2}{r^2} \right. \\ \left. + \frac{A^2 e^{\frac{2M}{r}}}{r} - \frac{2A^2 M e^{\frac{2M}{r}}}{r^2} \right]$$

Compare with pág. 74

$$F(r) = \frac{M}{r^2} - \frac{1}{r}$$

$$G(r) = \frac{M}{r^2} - \frac{1}{r} + \frac{A^2 e^{\frac{2M}{r}}}{r} - \frac{2A^2 M e^{\frac{2M}{r}}}{r^2}$$

Comprobación.

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$$\frac{t''}{t'} = -\frac{Mr'}{r^2}$$

$$\ln t' = \frac{M}{r} + \ln A$$

$$t' = A e^{\frac{M}{r}}$$

$$dt^2 = dx^2 - dy^2 - dz^2 = ds^2$$

$$\frac{dt}{ds} = 1$$

$$dt^2 [1 - v^2] = ds^2$$

$$\frac{dt}{ds} = \frac{1}{\sqrt{1 - v^2}}$$

La velocidad de la partícula
al infinito v_∞

$$\lim_{r \rightarrow \infty} A e^{\frac{M}{r}} = A$$

$$A = \frac{1}{\sqrt{1 - v_\infty^2}}$$

$$A = \frac{1}{\sqrt{1 - v_{\infty}^2}}$$

$$r'' = \left[-\frac{r'^2}{r} + \frac{A^2 e^{\frac{2M}{r}} - 1}{r} - \frac{M}{r^2} \right. \\ \left. - \frac{2M}{r^2} (A^2 e^{\frac{2M}{r}} - 1) + \frac{M r'^2}{r^2} \right]$$

$$r'' = \left[-\frac{r'^2}{r} - \frac{1}{r} + \frac{M}{r^2} + \frac{M r'^2}{r^2} \right. \\ \left. + \frac{A^2 e^{\frac{2M}{r}}}{r} - \frac{2MA^2}{r^2} e^{\frac{2M}{r}} \right]$$

Comprobado pág. 72

~~$r = 1$~~
Introducción de la nueva
variable $v = \frac{1}{r}$

$$r = \frac{1}{v}$$

$$r' = -\frac{v'}{v^2}$$

$$r'' = -\frac{v''}{v^2} + \frac{2v'^2}{v^3}$$

$$-\frac{v''}{v^2} + \frac{2v'^2}{v^3} = \left[-\left(\frac{v'^2}{v^4}\right)v - v + Mv^2 + M\frac{v'^2}{v^4}v^2 + vA^2e^{2Mv} - 2MA^2v^2e^{2Mv} \right]$$

$$-\frac{v''}{v^2} + \frac{2v'^2}{v^3} = \left[-\frac{v'^2}{v^3} - v + Mv^2 + \frac{Mv'^2}{v^2} + A^2ve^{2Mv} - 2MA^2v^2e^{2Mv} \right]$$

Principio Variacional para las
partículas exploradoras del
campo central de Birkhoff
en coordenadas polares esféricas.

$$\delta \int e^{\frac{2M}{r}} [1 + \dot{r}^2]$$

$$\left[\delta \int_A^B e^{\frac{2M}{r}} (1 + \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\varphi}^2) dt = 0 \right]$$

$$\left[\frac{\partial \mathcal{L}}{\partial r} = -\frac{2M}{r^2} e^{\frac{2M}{r}} [1 + v^2] \right. \\ \left. + 2e^{\frac{2M}{r}} [r \dot{\theta}^2 + r \sin^2 \theta \dot{\varphi}^2] \right]$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = 2e^{\frac{2M}{r}} \dot{r}$$

$$\left[\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} = -\frac{4M \dot{r}^2}{r^2} e^{\frac{2M}{r}} + 2e^{\frac{2M}{r}} \ddot{r} \right]$$

$$\cancel{2e^{\frac{2M}{r}} \ddot{r}} = \left[\cancel{\frac{2M \dot{r}^2}{r^2} e^{\frac{2M}{r}}} - \cancel{\frac{2M}{r^2} e^{\frac{2M}{r}}} [1+v^2] + \cancel{2e^{\frac{2M}{r}}} [r\ddot{\theta}^2 + r \cancel{\sin^2 \theta} \dot{\varphi}^2] \right] \quad 80$$

$$\ddot{r} = \left[\frac{2M \dot{r}^2}{r^2} - \frac{M}{r^2} [1+v^2] + r\ddot{\theta}^2 + r \sin^2 \theta \dot{\varphi}^2 \right]$$

$$\ddot{r} = \left[-\frac{\dot{r}^2}{r} + \frac{\dot{r}^2 + r^2 \ddot{\theta}^2 + r^2 \sin^2 \theta \dot{\varphi}^2}{r} + \frac{2M \dot{r}^2}{r^2} - \frac{M}{r^2} [1+v^2] \right]$$

$$\boxed{\ddot{r} = -\frac{\dot{r}^2}{r} + \frac{v^2}{r} + \frac{2M \dot{r}^2}{r^2} - \frac{M}{r^2} [1+v^2]}$$

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$$\frac{\partial \mathcal{L}}{\partial \theta} = 2e^{\frac{2M}{r}} r^2 \sin \theta \cos \theta \dot{\varphi}^2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 2e^{\frac{2M}{r}} r^2 \dot{\theta}$$

$$\left[\cancel{2} \cdot \left(-\frac{2M}{A^2} \dot{r} \right) e^{\frac{2M}{r}} A^2 \dot{\theta} + \cancel{2} e^{\frac{2M}{r}} r r \ddot{\theta} + \cancel{2} e^{\frac{2M}{r}} r^2 \ddot{\theta} \right] = \left[\cancel{2} e^{\frac{2M}{r}} r^2 \sin \theta \cos \theta \dot{\varphi}^2 \right]$$

$$r^2 \ddot{\theta} = \left[-2 r \dot{r} \dot{\theta} + r^2 \sin \theta \cos \theta \dot{\varphi}^2 + 2M \dot{r} \dot{\theta} \right]$$

$$\ddot{\theta} = \left[-2 \frac{\dot{r} \dot{\theta}}{r} + \sin \theta \cos \theta \dot{\varphi}^2 + 2M \frac{\dot{r} \dot{\theta}}{r^2} \right]$$

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$$\frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = 2 e^{\frac{2M}{r}} r^2 \sin^2 \theta \dot{\varphi}$$

$$\frac{d}{dt} \left(e^{\frac{2M}{r}} r^2 \sin^2 \theta \dot{\varphi} \right) = 0$$

$$\left. \begin{aligned} & - \frac{2M\dot{r}}{r^2} e^{\frac{2M}{r}} r^2 \sin^2 \theta \dot{\varphi} + 2 e^{\frac{2M}{r}} r \dot{r} \sin^2 \theta \dot{\varphi} \\ & + 2 e^{\frac{2M}{r}} r^2 \sin \theta \cos \theta \dot{\theta} \dot{\varphi} + e^{\frac{2M}{r}} r^2 \sin^2 \theta \ddot{\varphi} \end{aligned} \right\} = 0$$

$$\ddot{\varphi} = \frac{-2\dot{r}\dot{\varphi}}{r} - 2\dot{\theta}\dot{\varphi} \cot \theta + \frac{2M\dot{r}\dot{\varphi}}{r^2}$$

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$$t'' = - \frac{M t' r'}{r^2}$$

$$\frac{t''}{t'} = - \frac{M r'}{r^2}$$

$$\ln t' = + \frac{M}{r} + \ln A$$

$$t' = A e^{\frac{M}{r}}$$

$$r'' = \left[- \frac{r'^2}{r} + \frac{1}{r} A^2 e^{\frac{2M}{r}} - \frac{1}{r} + \frac{M}{r^2} - \frac{2M}{r^2} A^2 e^{\frac{2M}{r}} + \frac{M r'^2}{r^2} \right]$$

$$r = \frac{1}{v}$$

$$r' = - \frac{v'}{v^2}$$

$$\frac{dr}{dt} = \frac{dr}{dv} \frac{dv}{dt}$$

$$r'' = - \frac{v''}{v^2} + \frac{2v'^2}{v^3}$$

$$-\frac{v''}{v^2} + \frac{2v'^2}{v^3} = \left[\cancel{0} - \frac{v'^2 v}{v^4} \right.$$

$$+ A^2 v e^{2Mv} - v$$

$$\cancel{+ M v^2 - 2MA^2 v^2 e^{2Mv}} +$$

$$+ M v^2 - 2MA^2 v^2 e^{2Mv}$$

$$+ \frac{M v'^2 v^2}{v^4}$$

$$-\frac{v''}{v^2} + \frac{2v'^2}{v^3} = \left[-\frac{v'^2}{v^3} - v + A^2 v e^{2Mv} \right.$$

$$+ M v^2 - 2MA^2 v^2 e^{2Mv}$$

$$+ \frac{M v'^2 v}{v^2}$$

$$-UV'' + 3V'^2 = \left[-U^4 + A^2 U^4 e^{2Mu} \right. \\ \left. + MU^5 - 2MA^2 U^5 e^{2Mu} \right. \\ \left. + MUU'^2 \right]$$

$$\left. \begin{aligned} UV'' - 3V'^2 &= -U^4 + MU^5 + MUU'^2 \\ &+ A^2 U^4 e^{2Mu} - 2MA^2 U^5 e^{2Mu} \end{aligned} \right\} = 0$$

$$U = \alpha V$$

$$\cancel{s = \beta \sigma} \quad s = \beta \sigma$$

$$\cancel{\frac{2UV}{\beta^2} \frac{d^2 U}{d\sigma^2} - 3\alpha^2 \frac{dU}{d\sigma}}$$

$$\frac{\alpha^2}{\beta^2} U \frac{d^2 U}{d\sigma^2} - 3 \frac{\alpha^2}{\beta^2} \left(\frac{dU}{d\sigma} \right)^2 - \alpha^4 U^4$$

$$+ \alpha^5 M U^5 + \frac{\alpha^3}{\beta} M U \left(\frac{dU}{d\sigma} \right)^2$$

$$+ A^2 \alpha^4 U^4 e^{2M\alpha U} - 2MA^2 \alpha^5 U^5 e^{2M\alpha U}$$

$$2M\alpha = 2$$

$$\alpha = \frac{1}{M}$$

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$$0 = \left\{ \begin{aligned} & \frac{1}{M^2 \beta^2} U \frac{d^2 U}{d\sigma^2} - \frac{3}{M^2 \beta^2} \left(\frac{dU}{d\sigma} \right)^2 - \frac{1}{M^4} U^4 \\ & + \frac{1}{M^4} U^5 + \frac{1}{M^2 \beta^2} U \left(\frac{dU}{d\sigma} \right)^2 \\ & + \frac{A^2}{M^4} U^4 e^{2U} - \frac{2MA^2}{M^5} U^5 e^{2U} \end{aligned} \right.$$

$$0 = \left\{ \begin{aligned} & \frac{1}{M^2 \beta^2} U \frac{d^2 U}{d\sigma^2} - \frac{3}{M^2 \beta^2} \left(\frac{dU}{d\sigma} \right)^2 \\ & - \frac{1}{M^4} U^4 + \frac{1}{M^4} U^5 + \frac{1}{M^2 \beta^2} U \left(\frac{dU}{d\sigma} \right)^2 \\ & + \frac{A^2}{M^4} U^4 e^{2U} - \frac{2A^2}{M^4} U^5 e^{2U} \end{aligned} \right.$$

$$\beta = M$$

$$0 = \left\{ U \frac{d^2 U}{d\sigma^2} - 3 \left(\frac{dU}{d\sigma} \right)^2 - U^4 + U^5 \right. \\ \left. + U \left(\frac{dU}{d\sigma} \right)^2 + A^2 U^4 e^{2U} - 2A^2 U^5 e^{2U} \right\} \quad 87$$

E. Kamke. Gewöhnliche Differentialgleichungen.

Akademische Verlagsgesellschaft.
Leipzig. 1943. pag. 554. 1654

$$y'' + f(y) y'^2 + g(y) y' + h(y) = 0.$$

$$p(y) = y'(x)$$

$$\frac{dp}{dy} = \frac{y''}{y'} \quad \cancel{y'}$$

$$y'' = p \frac{dp}{dy}$$

$$p \frac{dp}{dy} + f(y) p^2 + g(y) p + h(y) = 0$$

$$UV'' - 3U'^2 - U^4 + U^5 + UV'^2 + A^2 U^4 e^{2U} - 2A^2 U e^{2U} \Bigg\} = 0 \quad 88$$

$$p(U) = U'(\sigma) = \frac{dU}{d\sigma}$$

$$\frac{dp}{dU} = \frac{\frac{dU'}{d\sigma}}{\frac{dU}{d\sigma}} = \frac{U''}{p}$$

$$U'' = p \frac{dp}{dU}$$

$$U p \frac{dp}{dU} - 3p^2 - U^4 + U^5 + U p^2 + A^2 U^4 e^{2U} - 2A^2 U e^{2U} \Bigg\} = 0$$

$$p \frac{dp}{dU} + \left[1 - \frac{3}{U} \right] p^2 + \left[U^4 - U^5 - 2A^2 U^4 e^{2U} + A^2 U^3 e^{2U} \right] = 0$$

$$\frac{dp}{dU} + \left[1 - \frac{3}{U}\right] p + \left[\frac{U^4 - U^3}{-2A^2 U^4 e^{2U}} + A^2 U^3 e^{2U} \right] \frac{1}{p} = 0$$

Bernoullische Differentialgleichung
Kamke, pag. 19.

4.5.

~~q~~ $q(U) = p^2 \quad \parallel \quad p = \sqrt{q}$

$$\frac{dq}{dU} = 2p \frac{dp}{dU}$$

$$\frac{dp}{dU} = \frac{1}{2} \frac{\frac{dq}{dU}}{\sqrt{q}}$$

$$\frac{1}{2} \frac{\frac{dq}{dT}}{\sqrt{q}} + \left[1 - \frac{3}{T}\right] \sqrt{q} + \left[\begin{array}{l} T^4 - T^3 \\ -2A^2 T^4 e^{2T} \\ + A^2 T^3 e^{2T} \end{array} \right] \frac{1}{\sqrt{q}} = 0$$

$$\left[\frac{dq}{dT} + \left[2 - \frac{6}{T}\right] q + \left[\begin{array}{l} T^4 - T^3 \\ -2A^2 T^4 e^{2T} \\ + A^2 T^3 e^{2T} \end{array} \right] \right] = 0$$

$$\frac{dq}{dT} + \left[2 - \frac{6}{T}\right] q = \left[\begin{array}{l} -T^4 + T^3 \\ + 2A^2 T^4 e^{2T} \\ - A^2 T^3 e^{2T} \end{array} \right]$$

$$f = 2 - \frac{6}{T}$$

$$F = \int \left[2 - \frac{6}{T}\right] dT$$

$$F = U - 6 \ln U \quad \parallel \quad -F = -U + 6 \ln U \quad 91$$

q2a
$$e^{-F} = U^6 e^{-U}$$

$$q = U^6 e^{-U} \left[C + \int U^6 e^{-U} \left\{ \begin{array}{l} -U^4 + U^3 \\ + 2A^2 U^4 e^{2U} \\ - A^2 U^3 e^{2U} \end{array} \right\} dU \right]$$

$$I_1 = - \int U^{10} e^{-U} dU$$

$$I_2 = \int U^9 e^{-U} dU$$

$$I_3 = 2A^2 \int U^{10} e^U dU$$

$$I_4 = -A^2 \int U^9 e^U dU$$

$$\cancel{I_c} = \cancel{f}$$

$$\frac{\ddot{\varphi}}{\dot{\varphi}} = -\frac{2\dot{r}}{r} - 2\dot{\theta}\cot\theta + \frac{2M\dot{r}}{r^2}$$

$$\ln \dot{\varphi} = -\ln r^2 - \ln \sin^2 \theta - \frac{2M}{r} + \ln A$$

$$\ln \left[r^2 \sin^2 \theta \dot{\varphi} e^{+\frac{2M}{r}} \right] = \ln C$$

$$e^{\frac{2M}{r}} r^2 \sin^2 \theta \dot{\varphi} = C$$

$$\int_A^B \tau_{ij} v^i v^j ds = 0$$

$$\int_A^B \tau_{ij} v^i v^j ds$$

$$\int_A^B \left[\sqrt{\tau_{ij} dx^i dx^j} + \sqrt{\tau_{ij} dx^i dx^j} \right] = 0$$

$$\frac{d}{ds} \left(\frac{\frac{\partial \tau_{ij}}{\partial x^k} \delta x^k dx^i dx^j + 2 \tau_{ij} \frac{\partial dx^i}{\partial x^k} dx^j}{2 d\tau} \right)$$

$$\frac{\frac{\partial \tau_{ij}}{\partial x^k} \delta x^k dx^i dx^j + 2 \tau_{ij} \frac{\partial dx^i}{\partial x^k} dx^j}{2 d\sigma}$$

$$\int_A^B \left(\frac{1}{2} \frac{\partial \tau_{ij}}{\partial x^k} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} d\tau \delta x^k + \frac{1}{2} \frac{\partial \tau_{ij}}{\partial x^k} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} d\sigma \delta x^k \right)$$

$$+ \left[\frac{d}{d\tau} \left\{ \tau_{ij} \frac{dx^j}{d\tau} + \tau_{kj} \frac{dx^j}{d\sigma} \right\} \right] \delta x^k$$

$$[2\alpha\eta_j - \delta_{ij}]$$

$$(\alpha V_i V_j - \delta_{ij})(\alpha V^i V^k - \beta \delta^{ik}) =$$

$$\begin{aligned} & \alpha^2 V_j V^k - \alpha\beta V^k V_j - \alpha\beta V_j V^k + \beta^2 \delta_j^k \\ & (\alpha^2 - 2\alpha\beta) V_j V^k + \beta^2 \delta_j^k \end{aligned}$$

$$\alpha^2 - 2\alpha\beta = 2$$

$$\beta^2 = -1$$

$$\beta = \pm i$$

$$\alpha^2 + 2i\alpha = 2$$

$$\alpha^2 + 2i\alpha - 2 = 0$$

$$\alpha = \frac{2i \pm \sqrt{-4 + 8}}{2}$$

$$\sqrt{\beta} = -1 + 2i$$

$$\alpha = i \pm 2$$

$$(\alpha = 2 + i) \quad \beta = i$$

$$\beta^2 = -1$$

$$\alpha^2 = 4 + 4i - 1 = 3 + 4i$$

$$-2\alpha\beta = +2 - 4i$$

$$[\alpha \bar{V}_i V_j - \beta \Delta_{ij}] [\alpha V^j V^k - \beta \Delta_{ij}^{ik}]$$

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$$= [\alpha \bar{V}_i V^k - \delta_i^k]$$

$$\alpha^2 \bar{V}_i V^k - \alpha \beta \bar{V}_i V^k - \alpha \beta \bar{V}_i V^k + \beta^2 \delta_i^k$$

$$(\alpha^2 - 2\alpha\beta) \bar{V}_i V^k + \beta^2 \delta_i^k$$

$$\alpha^2 - 2\alpha\beta = 2$$

$$\beta^2 = -1$$

$$\beta = \pm i \quad \beta_1 = +i$$

$$\alpha^2 - 2i\alpha - 2 = 0$$

$$\alpha = \frac{2i \pm \sqrt{4 + 8}}{2}$$

$$\alpha = i + 1$$

$$\alpha\beta = -1 + i$$

$$\alpha^2 = 2i \quad -2\alpha\beta = 2 - 2i$$

$$\boxed{\alpha^2 - 2\alpha\beta = 2}$$

$$\frac{\sqrt{M}[(1+i)V_i V_j - i \Delta_{ij}]}{\sqrt{\dot{V} \cdot \dot{V}}}$$

$$\delta \int \chi_{ij} \frac{dx^i}{d\tau} dx^j = 0$$

$$\delta \left[\frac{\partial \chi_{ij}}{\partial x^k} \delta x^k \frac{dx^i}{d\tau} dx^j + 2 \chi_{ik} \frac{dx^i}{d\tau} d(\delta x^k) \right]$$

~~$\partial \chi$~~

$$\frac{\partial \chi_{ij}}{\partial x^k} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} - 2 \frac{d}{d\tau} \left\{ \chi_{ik} \frac{dx^i}{d\tau} \right\} = 0$$

$$0 = \frac{\partial \chi_{ij}}{\partial x^k} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} - 2 \frac{\partial \chi_{ik}}{\partial x^j} \frac{dx^j}{d\tau} \frac{dx^i}{d\tau} - 2 \chi_{ik} \frac{d^2 x^i}{d\tau^2}$$

$$0 = \left[\chi^{kp} \frac{\partial \chi_{ij}}{\partial x^k} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} - 2 \chi^{kp} \frac{\partial \chi_{ik}}{\partial x^j} \frac{dx^j}{d\tau} \frac{dx^i}{d\tau} - 2 \frac{d^2 x^p}{d\tau^2} \right]$$

$$\left[\frac{d^2 x^i}{ds^2} \frac{dx^j}{ds} + \frac{d^2 x^j}{ds^2} \frac{dx^i}{ds} \right]$$

$$\frac{M [2V_i V_j - \Delta_{ij}] (M) [2V^i V^j - \Delta^{ij}]}{(V \cdot V)^2}$$

$$\frac{M^2}{(V \cdot V)^2} \quad \underline{4 - 2 - 2 + 4}$$

$$\sqrt{h^2} = \underline{h}$$

$$h_{ij} = h \delta_{ij}$$

$$\frac{\partial h_{ij}}{\partial x^k} = \frac{\partial h}{\partial x^k} \chi_{ij} + h \frac{\partial \chi_{ij}}{\partial x^k}$$

$$a_i = \left(\frac{\partial h}{\partial x^k} \chi_{ij} - \frac{\partial h}{\partial x^i} \chi_{jk} + h \frac{\partial \chi_{ij}}{\partial x^k} - h \frac{\partial \chi_{jk}}{\partial x^i} \right) \frac{dx^j}{ds} \frac{dx^k}{ds}$$

$$a_i = h \left[\frac{\partial h}{\partial x^k} \chi_{ij} - \frac{\partial h}{\partial x^i} \chi_{jk} + \frac{\partial \chi_{ij}}{\partial x^k} - \frac{\partial \chi_{jk}}{\partial x^i} \right] \frac{dx^j}{ds} \frac{dx^k}{ds}$$

Parámetro τ .

99

$$\boxed{t' = \frac{dt}{d\tau}}$$

$$\delta \int e^{\frac{2M}{r}} \left[1 + \left(\frac{dx}{dt} \right)^2 + \frac{dy}{dt} \right]$$

$$\delta \int e^{\frac{2M}{r}} (1 + v^2) dt = 0$$

$$\delta \int e^{\frac{2M}{r}} \left[1 + \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right] dt = 0$$

$$\delta \int e^{\frac{2M}{r}} \left[\cancel{1} + \frac{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2}{\left(\frac{dt}{d\tau} \right)^2} \right] \frac{dt}{d\tau} d\tau$$

$$\delta \int e^{\frac{2M}{r}} \left[t' + \frac{x'^2 + y'^2 + z'^2}{t'} \right] d\tau$$

Calcular las extremales del principio variacional:

$$\delta \int_A^B e^{\frac{2M}{r}} \left[t' + \frac{x'^2 + y'^2 + z'^2}{t'} \right] dt = 0$$

$$' = \frac{d}{dt}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

t — parámetro libre.

$$\frac{d}{dt} \left[e^{\frac{2M}{r}} \left[1 - \frac{x'^2 + y'^2 + z'^2}{t'^2} \right] \right] = 0$$

$$e^{\frac{2M}{r}} \frac{t'^2 - x'^2 - y'^2 - z'^2}{t'^2} = C$$

$$e^{\frac{2M}{r}} = C \left(\frac{dt}{ds} \right)^2$$

$$\frac{dt}{ds} = A e^{\frac{M}{r}}$$

$$\frac{\partial \mathcal{L}}{\partial x} = - \frac{2Mx}{r^3} e^{\frac{2M}{r}} \left[t' + \frac{x'^2 + y'^2 + z'^2}{t'} \right] \quad 101$$

$$\frac{\partial \mathcal{L}}{\partial x'} = e^{\frac{2M}{r}} \left[\frac{2x'}{t'} \right]$$

$$0 = \frac{d}{dt} \left[e^{\frac{2M}{r}} \frac{2x'}{t'} \right] + \frac{2Mx}{r^3} e^{\frac{2M}{r}} \left[t' + \frac{x'^2 + y'^2 + z'^2}{t'} \right]$$

$$0 = \left[- \frac{2M}{r^2} \frac{2x'}{t'} e^{\frac{2M}{r}} + \frac{2Mx}{r^3} e^{\frac{2M}{r}} \right]$$

$$0 \neq e^{\frac{2M}{r}}$$

$$0 = \left[- \frac{2M}{r^2} e^{\frac{2M}{r}} \frac{2x'}{t'} + e^{\frac{2M}{r}} \frac{2x''}{t'} - e^{\frac{2M}{r}} \frac{2x' t''}{t'^2} + e^{\frac{2M}{r}} \frac{2Mx}{r^3} \left[\frac{t'^2 + x'^2 + y'^2 + z'^2}{t'} \right] \right]$$

$$x'' = \left[+ \frac{2Mx'r'}{r^2} + \frac{x't''}{t'} - \frac{Mx}{r^3} \left[t'^2 + x'^2 + y'^2 + z'^2 \right] \right]$$

$$x'' = -\frac{Mx}{r^3} - \frac{2Mx}{r^3} (x'^2 + y'^2 + z'^2) + \frac{Mx't'}{r^2}$$

$$\frac{t''}{t'} = -\frac{M}{r^2}$$

Calcular los extremos del 103
principio variacional

$$\delta \int_A^B e^{\frac{2M}{r}} t'^2 dt$$

$$\delta \int_H^B e^{\frac{2M}{r}} \left[(1-\lambda) t'^2 + (1-\lambda) [x'^2 + y'^2 + z'^2] \right] dt$$

$$\delta \int_A^B e^{\frac{2M}{r}} dt$$

Calcular las extremales del¹⁰⁴
principio variacional:

$$\delta \int_A^B e^{\frac{2M}{r}} \left[(1+\lambda)t' + (1-\lambda) \frac{x'^2 + y'^2 + z'^2}{t'} \right] dt$$

$$\boxed{\prime = \frac{d}{dt}}$$

λ -Multiplicador
de Lagrange

$$\frac{\partial \mathcal{L}}{\partial t'} = e^{\frac{2M}{r}} \left[(1+\lambda) - (1-\lambda) \frac{x'^2 + y'^2 + z'^2}{t'^2} \right]$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial t'} = \frac{d}{dt} \left[e^{\frac{2M}{r}} \left\langle (1+\lambda) - (1-\lambda) \frac{x'^2 + y'^2 + z'^2}{t'^2} \right\rangle \right]$$

$$\boxed{t'^2 - x'^2 - y'^2 - z'^2 = 1}$$

$$\frac{d}{dt} \left[e^{\frac{2M}{r}} \left(\cancel{\frac{1}{t'^2}} + \lambda \right) \right]$$

$$\frac{d}{dt} \left[e^{\frac{2M}{r}} \left\langle (1+\lambda)t'^2 - (1-\lambda)[t'^2 - 1] \right\rangle \right] = 0$$

$$e^{\frac{2M}{r}} \left\langle 2\lambda t'^2 - (1-\lambda) \right\rangle = A$$

$$e^{\frac{2M}{r}} \left\langle 1 + \frac{1}{t^{1/2}} \right\rangle = C$$

$$\oint_A^B e^{\frac{2M}{r}} \left[H \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right] dt = 0$$

$$S = e^{\frac{2M}{r}} (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)$$

$$p_x = 2e^{\frac{2M}{r}} \dot{x}; \quad p_y = 2e^{\frac{2M}{r}} \dot{y}; \quad p_z = 2e^{\frac{2M}{r}} \dot{z}$$

$$2e^{\frac{2M}{r}} \dot{x} = \frac{-2Mx}{r^3} (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)$$

$$2e^{\frac{2M}{r}} \dot{y} = -2M$$

~~$\oint e^{\frac{2M}{r}}$~~

$$p_x = 2e^{\frac{2M}{r}} \dot{x}$$

$$p_y = 2e^{\frac{2M}{r}} \dot{y}$$

$$p_z = 2e^{\frac{2M}{r}} \dot{z}$$

$$H = \left[p_x \dot{x} + p_y \dot{y} + p_z \dot{z} \right] - e^{\frac{2M}{r}}$$

$$\dot{x} = \frac{1}{2} p_x e^{-\frac{2M}{r}}$$

$$\dot{y} = \frac{1}{2} p_y e^{-\frac{2M}{r}}$$

$$\dot{z} = \frac{1}{2} p_z e^{-\frac{2M}{r}}$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \frac{1}{4} p^2 e^{-\frac{4M}{r}}$$

$$H = \left[p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - e^{-\frac{2M}{r}} - \frac{1}{4} p^2 e^{-\frac{2M}{r}} \right]$$

~~$$\dot{x} = \dot{x} - \frac{1}{4} p_x e^{-\frac{2M}{r}}$$~~

$$p_x \dot{x} + p_y \dot{y} + p_z \dot{z} = \frac{1}{2} p^2 e^{-\frac{2M}{r}}$$

$$H = \frac{1}{2} p^2 e^{-\frac{2M}{r}} - e^{-\frac{2M}{r}} - \frac{1}{4} p^2 e^{-\frac{2M}{r}}$$

$$H = \frac{1}{4} p^2 e^{-\frac{2M}{r}} - e^{-\frac{2M}{r}} \quad \text{Comprobado}$$

$$\int_A^B e^{\frac{2M}{r}} [1 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2] dt$$

$$L = e^{\frac{2M}{r}} [1 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2]$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = 2\dot{x} e^{\frac{2M}{r}} \quad \Bigg| \quad \dot{x} = \frac{1}{2} p_x e^{-\frac{2M}{r}}$$

$$p_y = 2\dot{y} e^{\frac{2M}{r}} \quad \Bigg| \quad \dot{y} = \frac{1}{2} p_y e^{-\frac{2M}{r}}$$

$$p_z = 2\dot{z} e^{\frac{2M}{r}} \quad \Bigg| \quad \dot{z} = \frac{1}{2} p_z e^{-\frac{2M}{r}}$$

$$\dot{x} p_x + \dot{y} p_y + \dot{z} p_z = 2 e^{\frac{2M}{r}} [\dot{x}^2 + \dot{y}^2 + \dot{z}^2]$$

$$\dot{x} p_x + \dot{y} p_y + \dot{z} p_z = \frac{2 e^{\frac{2M}{r}}}{4} (p_x^2 + p_y^2 + p_z^2) e^{-\frac{4M}{r}}$$

$$\dot{x} p_x + \dot{y} p_y + \dot{z} p_z = \frac{1}{2} e^{-\frac{2M}{r}} (p_x^2 + p_y^2 + p_z^2)$$

$$L = e^{\frac{2M}{r}} + \frac{e^{\frac{2M}{r}}}{4} [p_x^2 e^{-\frac{4M}{r}} + p_y^2 e^{-\frac{4M}{r}} + p_z^2 e^{-\frac{4M}{r}}]$$

$$L = e^{\frac{2M}{r}} + \frac{1}{4} p^2 e^{-\frac{2M}{r}}$$

Comprobado

$$H = \frac{1}{4} p^2 e^{-\frac{2M}{r}} \quad \text{---} \quad e^{\frac{2M}{r}}$$

$$\bar{W} = e^{\frac{M}{r}} [\beta_1^2 + \beta_2^2 + \beta_3^2]$$

$$p_x = -\frac{Mx}{r^3} \beta^2 e^{\frac{M}{r}} \quad \alpha_1 = -2\beta_1 e^{\frac{M}{r}}$$

$$p_y = -\frac{My}{r^3} \beta^2 e^{\frac{M}{r}} \quad \alpha_2 = -2\beta_2 e^{\frac{M}{r}}$$

$$p_z = -\frac{Mz}{r^3} \beta^2 e^{\frac{M}{r}} \quad \alpha_3 = -2\beta_3 e^{\frac{M}{r}}$$

$$\alpha^2 = 4\beta^2 e^{\frac{2M}{r}}$$

$$e^{\frac{2M}{r}} = \frac{\alpha^2}{4\beta^2}$$

~~$$e^{\frac{2M}{r}} = \frac{4\beta^2}{\alpha^2}$$~~

~~$$p^2 = \frac{M^2}{r} \beta^4 e^{\frac{M}{r}}$$~~

$$p^2 = \frac{M^2}{r} \beta^4 e^{\frac{2M}{r}}$$

$$\frac{1}{4} p^2 e^{-\frac{2M}{r}} = \frac{1}{4} \frac{M^2}{r} \beta^4$$

$$H = \frac{1}{4} p^2 e^{-\frac{2M}{r}} - e^{\frac{2M}{r}}$$

$$S = e^{\frac{2M}{r}} (\alpha_1^2 + \alpha_2^2 + \alpha_3^2) = e^{\frac{2M}{r}} \alpha^2$$

$$p_x = -\frac{2Mx}{r^3} e^{\frac{2M}{r}} \alpha^2$$

$$p^2 = -\frac{2M}{r} e^{\frac{4M}{r}} \alpha^4$$

$$p_y = -\frac{2My}{r^3} e^{\frac{2M}{r}} \alpha^2$$

$$p_z = -\frac{2Mz}{r^3} e^{\frac{2M}{r}} \alpha^2$$

$$\beta_1 = 2\alpha_1 e^{\frac{2M}{r}}$$

$$\beta_2 = 2\alpha_2 e^{\frac{2M}{r}}$$

$$\beta_3 = 2\alpha_3 e^{\frac{2M}{r}}$$

$$H = \frac{1}{4} p^2 e^{-\frac{2M}{r}} - e^{\frac{2M}{r}}$$

$$W = e^{\frac{M}{r}} [\beta_1^2 + \beta_2^2 + \beta_3^2]$$

$$p_x = -\frac{Mx}{r^3} \beta^2 e^{\frac{M}{r}} \quad \alpha_1 = -2\beta_1 e^{\frac{M}{r}}$$

$$p_y = -\frac{My}{r^3} \beta^2 e^{\frac{M}{r}} \quad \alpha_2 = -2\beta_2 e^{\frac{M}{r}}$$

$$p_z = -\frac{Mz}{r^3} \beta^2 e^{\frac{M}{r}} \quad \alpha_3 = -2\beta_3 e^{\frac{M}{r}}$$

$$p^2 = \frac{M^2}{r} \beta^4 e^{\frac{2M}{r}} \quad \alpha^2 = 4\beta^2 e^{\frac{2M}{r}}$$

$$e^{\frac{2M}{r}} = \frac{\alpha^2}{4\beta^2}$$

$$H = \frac{1}{4} \frac{M^2}{r} \beta^4 - \frac{\alpha^2}{4\beta^2}$$

$$\eta = e^{-\frac{M}{r}} \quad y$$

$$\xi = e^{-\frac{M}{r}} \quad z$$

$$\cancel{\eta = e^{-\frac{2M}{r}}}$$

$$e^{\frac{2M}{r}}$$

$$\left[e^{\frac{2M}{r}} + e^{\frac{2M}{r}} v^2 \right] dt$$

$$\xi = e^{-\frac{M}{r}} \quad \eta = e^{-\frac{M}{r}}$$

$$y = e^{-\frac{M}{r}} \eta$$

$$z = e^{-\frac{M}{r}} \xi$$

$$r^2 = e^{-\frac{2M}{r}} \rho^2$$

$$\cancel{\dot{x} = -\frac{M}{r^2} \xi + e^{-\frac{M}{r}} \dot{\xi}}$$

$$\cancel{v^2 = + \frac{M^2}{r^4} \xi^2 + e^{-\frac{2M}{r}}}$$

$$x = e^{\frac{\alpha}{r}}$$

$$y = e^{\frac{\alpha}{r}} \eta$$

$$z = e^{\frac{\alpha}{r}} \rho$$

$$r^2 = e^{\frac{2\alpha}{r}} \rho^2$$

$$r = e^{\frac{\alpha}{r}} \rho$$

$$\dot{x} = -\frac{\alpha \dot{r}}{r^2} e^{\frac{\alpha}{r}} + e^{\frac{\alpha}{r}} \dot{r}$$

$$v^2 = e^{\frac{2\alpha}{r}} \left[\frac{\alpha^2 \dot{r}^2}{r^4} \rho^2 + \dot{\rho}^2 - \frac{2\alpha \dot{r} \rho \dot{\rho}}{r^2} \right]$$

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$$e^{\frac{2M}{r}} v^2 = e^{\frac{2M+2\alpha}{r}} \left[\frac{\alpha^2 \dot{r}^2}{r^4} \rho^2 + \dot{\rho}^2 - \frac{2\alpha \dot{r} \rho \dot{\rho}}{r^2} \right]$$

$$8\pi \rho_0 [\vec{V}_i \cdot \vec{V}_j] - \square h_{ij} = \frac{1}{2} \Delta_{ij}$$

$$\frac{\partial A_{ij}}{\partial x^p} \frac{dx^i}{dt} \frac{dx^j}{dt} - 2 \frac{\partial A_{pj}}{\partial x^i} \frac{dx^i}{dt} \frac{dx^j}{dt} - 2 A_{pm} \Delta_{mq} \left(\frac{\partial h_{ic}}{\partial x^j} - \frac{\partial h_{ij}}{\partial x^c} \right) \frac{dx^i}{ds} \frac{dx^j}{ds}$$

~~$$\frac{\partial A_{ij}}{\partial x^p} - 2 \frac{\partial A_{pj}}{\partial x^i} - 2 A_{pm} \Delta_{mq} \left(\frac{\partial h_{ic}}{\partial x^j} - \frac{\partial h_{ij}}{\partial x^c} \right) = 0$$~~

$$\frac{\partial A_{ij}}{\partial x^p} - 2 \frac{\partial A_{pj}}{\partial x^i} - 2 A_{pm} \Delta_{mq} \left(\frac{\partial h_{ic}}{\partial x^j} - \frac{\partial h_{ij}}{\partial x^c} \right) = 0$$

$$a^{pr} \frac{\partial A_{ij}}{\partial x^p} - 2 a^{pr} \frac{\partial A_{pj}}{\partial x^i} - 2 \Delta^{rm} \left(\frac{\partial h_{ic}}{\partial x^j} - \frac{\partial h_{ij}}{\partial x^c} \right) = 0$$

$$\Delta^{rq} \left(\frac{\partial h_{ic}}{\partial x^j} - \frac{\partial h_{ij}}{\partial x^c} \right) = \frac{1}{2} a^{pr} \frac{\partial A_{ij}}{\partial x^p} - a^{pr} \frac{\partial A_{pj}}{\partial x^i}$$

$$\oint_A \left[A_{ij} \frac{dx^i}{dt} dx^j + A_{ijkl} \frac{dx^i}{dt} \frac{dx^j}{dt} \frac{dx^k}{dt} \frac{dx^l}{dt} + A_{ijklmn} \frac{dx^i}{dt} \frac{dx^j}{dt} \frac{dx^k}{dt} \frac{dx^l}{dt} \frac{dx^m}{dt} \frac{dx^n}{dt} \right]$$

~~$$\oint_A \left[A_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right]$$~~

$$\oint_A \left[A_{ij} \frac{dx^i}{dt} dx^j \right] = 0$$

$$\oint_A \left[\frac{\partial A_{ij}}{\partial x^p} \delta x^p + 2A_{pj} \cdot \frac{d}{dt} (\delta x^p) \frac{dx^j}{dt} \right] = 0$$

$$\frac{\partial A_{ij}}{\partial x^p} \frac{d}{dt} \left(\frac{dx^i}{dt} \frac{dx^j}{dt} \right) = 0$$

$$\frac{\partial A_{ij}}{\partial x^p} - 2 \frac{\partial A_{pj}}{\partial x^i}$$

$$\left[\frac{\partial A_{ij}}{\partial x^p} \frac{dx^i}{dt} \frac{dx^j}{dt} - 2 \frac{\partial A_{pj}}{\partial x^i} \frac{dx^i}{dt} \frac{dx^j}{dt} - 2 A_{pj} \frac{dx^j}{dt} \right]$$

$$a^{ps} \frac{\partial A_{ij}}{\partial x^p} - a^{ps} \frac{\partial A_{ip}}{\partial x^j} - a^{ps} \frac{\partial A_{ji}}{\partial x^i} - 2 \Delta^{st} \left(\frac{\partial h_{ri}}{\partial x^s} - \frac{\partial h_{ij}}{\partial x^r} \right) = 0$$

$$a^{ps} \frac{\partial A_{ij}}{\partial x^p} - 2 a^{ps} \frac{\partial A_{pi}}{\partial x^j} - 2 \Delta^{ps} \frac{\partial h_{pi}}{\partial x^j} + 2 \Delta^{ps} \frac{\partial h_{ij}}{\partial x^p} = 0$$

$$a^{ps} \frac{\partial A_{ij}}{\partial x^p} = -2 \Delta^{ps} \frac{\partial h_{ij}}{\partial x^p}$$

$$a^{ps} \frac{\partial A_{pi}}{\partial x^j} = -\Delta^{ps} \frac{\partial h_{pi}}{\partial x^j}$$

$$a^{ps} A_{ij}$$

$$a^{ps} \frac{\partial}{\partial x^p}$$

$$a^{ps} \frac{\partial A}{\partial x^p} = -2 \Delta^{ps} \frac{\partial h}{\partial x^p}$$

$$A_{st} a^{ps} \frac{\partial A}{\partial x^p} = -A_{st} \Delta^{ps} \frac{\partial h}{\partial x^p}$$

$$\frac{\partial A}{\partial x^t} = -A_{st} \Delta^{ps} \frac{\partial h}{\partial x^p}$$

Principio Variacional.

$$\delta \int_A^b A_{ij} \frac{dx^i}{dt} dx^j = 0$$

τ parámetro arbitrario.

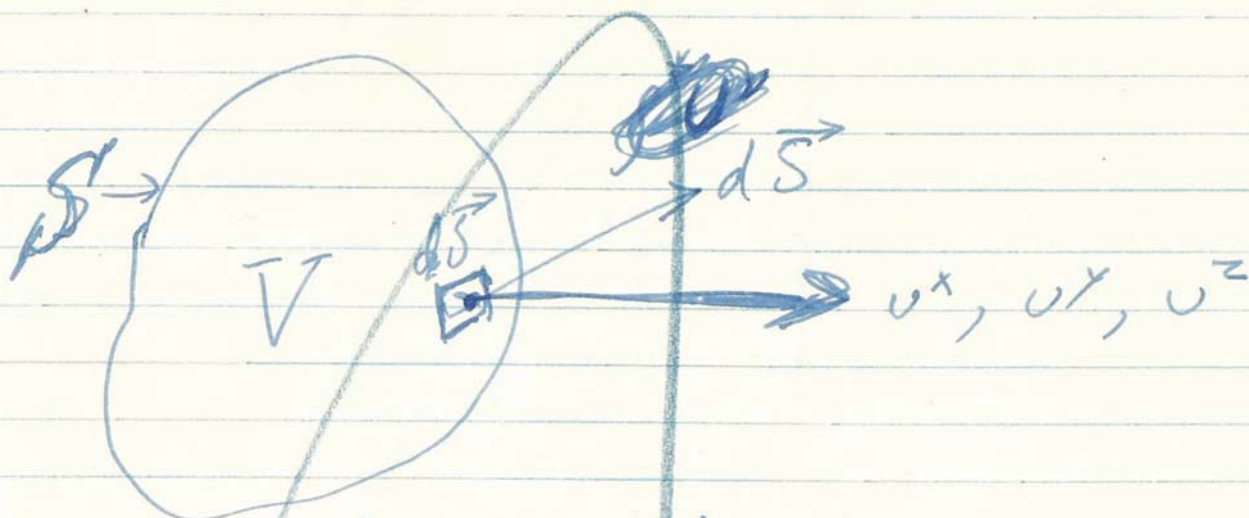
$$\int_A^b \left[\frac{\partial A_{ij}}{\partial x^p} \frac{dx^i}{dt} \frac{dx^j}{dt} \delta x^p dt - \frac{d}{dt} \left\{ A_{ip} \frac{dx^i}{dt} + A_{pj} \frac{dx^j}{dt} \right\} \delta x^p dt \right] = 0$$

$$\begin{aligned} \frac{\partial A_{ij}}{\partial x^p} \frac{dx^i}{dt} \frac{dx^j}{dt} - \frac{\partial A_{ip}}{\partial x^j} \frac{dx^i}{dt} \frac{dx^j}{dt} - \frac{\partial A_{pj}}{\partial x^i} \frac{dx^i}{dt} \frac{dx^j}{dt} \\ - A_{ip} \frac{d^2 x^i}{dt^2} - A_{pj} \frac{d^2 x^j}{dt^2} \end{aligned} = 0$$

$$\frac{\partial A_{ij}}{\partial x^p} - \frac{\partial A_{ip}}{\partial x^j} - \frac{\partial A_{ji}}{\partial x^i} - A_{qp} \Delta^q \left(\frac{\partial h_{ri}}{\partial x^j} - \frac{\partial h_{ij}}{\partial x^r} \right) - A_{pq} \Delta^q \left(\frac{\partial h_{ri}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^r} \right)$$

$$T^{\mu\nu} = \frac{1}{c^2} \rho^* v^\mu v^\nu$$

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mass tensor.



Ecuación de continuidad

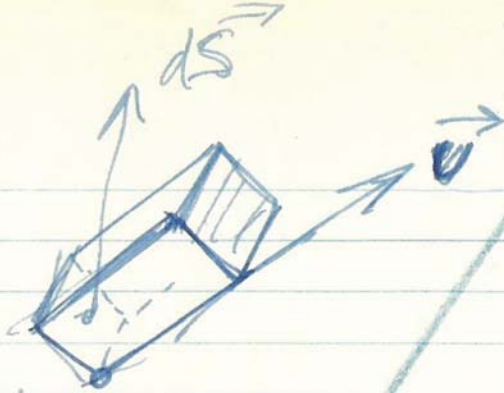
A diagram of a small rectangular volume element dV with a differential surface element $d\vec{S}$ on one of its faces. The equation $dV = (\vec{v} dt) \cdot d\vec{S}$ is written next to it.

$$dV = (\vec{v} dt) \cdot d\vec{S}$$

$$\rho^* (\vec{v} dt) \cdot d\vec{S} = dm$$

$$\oint_S \rho^* (\vec{v} dt) \cdot d\vec{S} = \iiint_V \text{div}(\rho^* \vec{v}) dV dt$$

$$\frac{\partial(\rho^* v^x)}{\partial x} + \frac{\partial(\rho^* v^y)}{\partial y} + \frac{\partial(\rho^* v^z)}{\partial z} = -\frac{\partial \rho^*}{\partial t}$$



Cantidad de materia contenida
en el volumen elemento de
volumen.

$$\vec{v} \cdot d\vec{S} dt$$

Elemento de
volumen para
el observador O

$$\vec{v} \cdot d\vec{S} dt$$

Elemento de
volumen ~~para~~
en el sistema
en reposo

$$\frac{\vec{v} \cdot d\vec{S} dt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

~~Cantidad de~~
masa contenida
en el elemento
de volumen

$$\rho^* \frac{\vec{v} \cdot d\vec{S} dt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Cantidad de materia que sale
en dt segundos, del volumen ΔV

$$\int_{\Delta V} \rho^* \frac{\vec{v} \cdot d\vec{S} dt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \frac{d\rho^*}{dt} dV$$

$$\frac{dV}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{elemento de volumen en reposo}$$

$$\frac{\partial \rho^*}{\partial t} \cdot \frac{dV}{\sqrt{1 - \frac{v^2}{c^2}}} dt \quad \text{aumento de materia en } dt \text{ segundos en el elemento de volumen } V$$



- 1) Para establecer la ecuación de continuidad se considera un ~~elemento de~~ volumen ΔV en reposo para el observador O . El volumen ΔV está ~~rodeado de una~~ limitado por una superficie ΔS en reposo para O .
- 2) ΔV está sumergido en un cuerpo continuo en movimiento.

$$\square h_{jk} + \frac{4}{3}\pi \rho^* \Delta_{jk} = 8\pi T_{jk}$$

$$T_{jk} = \rho^* V_j V_k$$

$$T_{jk} \quad T^{jk} = \rho^* V^j V^k$$

$$\frac{\partial}{\partial x^k} T^{jk} = \underbrace{\frac{\partial \rho^*}{\partial x^k} V^j}_{\text{eq. por continuidad}} + \rho^* V^k \frac{\partial}{\partial x^k} V^j$$

$$\boxed{\frac{\partial}{\partial x^k} T^{jk} = \rho^* A^j}$$



~~$\frac{\partial}{\partial t} T^{jk}$~~

$$\square h^{jk} + 4\pi \rho^* \Delta^{jk} = 8\pi T^{jk}$$

$$\square \frac{\partial h^{jk}}{\partial x^k} + 4\pi \frac{\partial \rho^*}{\partial x^k} \Delta^{jk} = 8\pi \frac{\partial}{\partial x^k} T^{jk}$$

$$\delta \int_A^B \frac{A_{ij} dx^i dx^j}{V_k dx^k} = 0$$

$$\delta \int_A^B \left[\frac{\frac{\partial A_{ij}}{\partial x^p} dx^i dx^j}{V_k dx^k} \delta x^p \right]$$

$$= \frac{A_{ij} dx^i dx^j}{(V_k dx^k)^2} \left[\frac{\partial V_k dx^k}{\partial x^p} \right] \delta x^p$$

$$+ \frac{2 A_{ip} dx^i d(\delta x^p)}{V_k dx^k}$$

$$- \frac{A_{ij} dx^i dx^j}{(V_k dx^k)^2} V_p d(\delta x^p)$$

$$\frac{\frac{\partial A_{ij}}{\partial x^p} dx^i dx^j}{V_k dx^k ds} = \frac{A_{ij} dx^i dx^j}{(V_k dx^k)^2} \frac{\partial V_k}{\partial x^p} \frac{dx^p}{ds}$$

$$= \frac{d}{ds} \left[\frac{A_{ip} dx^i}{V_k dx^k} \right]$$

$$= \frac{d}{ds} \left[\frac{A_{ij} dx^i dx^j}{(V_k dx^k)^2} V_p \right]$$

$$\nabla_{\beta} A^{\gamma} = \frac{\partial A^{\gamma}}{\partial x^{\beta}} + \Gamma_{\alpha\beta}^{\gamma} A^{\alpha}$$

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$$\nabla_{\beta} A_{\alpha} = \frac{\partial A_{\alpha}}{\partial x^{\beta}} - \Gamma_{\alpha\beta}^{\gamma} A_{\gamma}$$

$$\nabla_{\beta} T^{\mu\nu} = \frac{\partial T^{\mu\nu}}{\partial x^{\beta}} + \Gamma_{\beta\alpha}^{\mu} T^{\alpha\nu} + \Gamma_{\beta\alpha}^{\nu} T^{\mu\alpha}$$

$$\nabla_{\beta} T^{\mu\nu} = \frac{\partial T^{\mu\nu}}{\partial x^{\beta}} + \Gamma_{\beta\sigma}^{\mu} T^{\sigma\nu} + \Gamma_{\beta\sigma}^{\nu} T^{\mu\sigma}$$

$$\nabla_{\nu} T^{\mu\nu} = \frac{\partial T^{\mu\nu}}{\partial x^{\nu}} + \Gamma_{\nu\sigma}^{\mu} T^{\sigma\nu} + \Gamma_{\nu\sigma}^{\nu} T^{\mu\sigma}$$

$$\Gamma_{\nu\sigma}^{\nu} = g^{\nu\alpha} \left(\frac{\partial g_{\nu\alpha}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\nu\sigma}}{\partial x^{\alpha}} \right)$$

$$\Gamma_{\nu\sigma}^{\nu} = g^{\nu\alpha} \frac{\partial g_{\nu\alpha}}{\partial x^{\sigma}}$$

$$g^{\nu\alpha} g_{\nu\alpha} = \delta^{\nu}_{\nu}$$

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{vmatrix}$$

$$g g^{\mu\nu}$$

~~dg~~

$$\cancel{dg = \frac{\partial g}{\partial g_{\mu\nu}} (g^{\mu\nu} g) dg_{\mu\nu}}$$

$$dg = (g g^{\mu\nu}) dg_{\mu\nu}$$

$$\frac{\partial g}{\partial x^\alpha} = g g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\alpha}$$

$$\Gamma^\alpha_{\alpha\gamma} = g^{\alpha\mu} \left[\frac{\partial g_{\alpha\mu}}{\partial x^\gamma} + \frac{\partial g_{\gamma\mu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\gamma}}{\partial x^\mu} \right]$$

$$\Gamma^\alpha_{\alpha\gamma} = g^{\alpha\mu} \frac{\partial g_{\alpha\mu}}{\partial x^\gamma} = \frac{1}{2g} \frac{\partial g}{\partial x^\gamma}$$

$$\Gamma^\alpha_{\alpha\gamma} = \frac{\partial}{\partial x^\gamma} \ln \sqrt{-g}$$

~~$\nabla_\mu A^\mu$~~

$$\log \sqrt{-g} = +\frac{1}{2} \log g + C$$

$$\nabla_\nu A^\nu = \frac{\partial A^\nu}{\partial x^\nu} + \Gamma^\nu_{\nu\alpha} A^\alpha$$

$$\nabla_\nu A^\nu = \frac{\partial A^\nu}{\partial x^\nu} + A^\alpha \frac{\partial}{\partial x^\alpha} \log \sqrt{-g}$$

$$\nabla_\nu A^\nu = \frac{\partial A^\alpha}{\partial x^\alpha} + \frac{A^\alpha}{2g} \frac{\partial g}{\partial x^\alpha}$$

$$\frac{\partial}{\partial x^\alpha} \{ \sqrt{-g} A^\alpha \} = \sqrt{-g} \frac{\partial A^\alpha}{\partial x^\alpha} + \frac{-\frac{\partial g}{\partial x^\alpha} A^\alpha}{2\sqrt{-g}}$$

$$\frac{\partial}{\partial x^\alpha} \{ \sqrt{-g} A^\alpha \} = \sqrt{-g} \sqrt{-g} \left(\frac{\partial A^\alpha}{\partial x^\alpha} + \frac{A^\alpha \frac{\partial g}{\partial x^\alpha}}{2g} \right)$$

$$\frac{\partial}{\partial x^\alpha} \{ \sqrt{-g} A^\alpha \} = \sqrt{-g} \nabla_\alpha A^\alpha$$

$$\nabla_\alpha A^\alpha = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \{ \sqrt{-g} A^\alpha \}$$

$$t'' = -\frac{Mt'r'}{r^2}$$

$$x'' = \frac{Mx}{r^3} - \frac{2Mx}{r^3}(t'^2 - 1) + \frac{Mt'r'}{r^2}$$

~~$\frac{d}{dr}$~~ $\frac{d}{ds} = r' \frac{d}{dr}$

~~$\frac{d^2}{ds^2} = r' \frac{d}{dr}$~~

$$\frac{d^2}{ds^2} = r' \frac{d}{dr} \left[r' \frac{d}{dr} \right]$$

$$\frac{d^2}{ds^2} = r' \left[r' \frac{d^2}{dr^2} + r'' \frac{d}{dr} \right]$$

$$\frac{dx}{dt} = \frac{\left(\frac{dx}{dr}\right) \left(\frac{dr}{dt}\right)}{\left(\frac{dt}{dt}\right)}$$

$$\frac{d^2x}{dt^2} = \frac{1}{\left(\frac{dt}{dr}\right)} \frac{\frac{dr}{dt} \frac{d^2x}{dr^2} - \frac{dx}{dr} \frac{d^2t}{dr^2}}{\left(\frac{dt}{dr}\right)^2}$$

$$\frac{dx}{dr} = \frac{\left(\frac{dx}{dt}\right)}{\left(\frac{dr}{dt}\right)} = \frac{\dot{x}}{\dot{r}}$$

$$\frac{d^2x}{dr^2} = - \frac{\left(\frac{dx}{dt}\right)}{\dot{r} \left(\frac{dr}{dt}\right)^2} \frac{d^2r}{dt^2} + \frac{\frac{d^2x}{dt^2}}{\dot{r} \left(\frac{dr}{dt}\right)}$$

$$\frac{d^2x}{dr^2} = - \frac{\dot{x}}{\dot{r}^3} \ddot{r} + \frac{\ddot{x}}{\dot{r}^2}$$

$$\frac{dx}{dr} = \frac{\dot{x}}{\dot{r}} \quad \parallel \quad \frac{d}{dr} = \frac{1}{\dot{r}} \frac{d}{dt}$$

$$\frac{d^2x}{dr^2} = \frac{1}{\dot{r}} \frac{d}{dt} \frac{\dot{x}}{\dot{r}}$$

$$\frac{d^2x}{dr^2} = \frac{1}{\dot{r}} \frac{\dot{r} \ddot{x} - \dot{x} \ddot{r}}{\dot{r}^2}$$

$$\frac{d^2x}{dr^2} = \frac{\ddot{x}}{\dot{r}^2} - \frac{dx}{dr} \frac{\ddot{r}}{\dot{r}^2}$$

$$\frac{d^2x}{dr^2} = \frac{M}{\dot{r}^4 r^3} \left[-x - x v^2 + 2r \dot{x} \dot{r} \right] - \frac{dx}{dr} \frac{\ddot{r}}{\dot{r}^2}$$

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$\frac{v^2}{r^2} = \left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2$$

$$\frac{\ddot{x}}{r^2} = - \frac{Mx}{r^3} \left[\left(\frac{dt}{dr}\right)^2 + \left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2 \right] + \frac{2Mr}{r^3} \left(\frac{dx}{dr}\right)$$

$$\frac{\ddot{r}}{r^2} = - \frac{1}{r} + \frac{1}{r} \left[\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2 \right] - \frac{M}{r^2} \left[\left(\frac{dt}{dr}\right)^2 + \left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2 \right] + \frac{2M}{r^2}$$

$$= \left[- \frac{Mx}{r^3} \left(\frac{dt}{dr}\right)^2 + \frac{Mx}{r^3} \left[\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2 \right] + \frac{2M}{r^2} \frac{dx}{dr} + \frac{1}{r} - \frac{1}{r} \left[\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2 \right] + \frac{M}{r^2} \frac{dx}{dr} \left[\left(\frac{dt}{dr}\right)^2 + \left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2 \right] \right]$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$1 = \frac{x \left(\frac{dx}{dr} \right) + y \left(\frac{dy}{dr} \right) + z \left(\frac{dz}{dr} \right)}{r}$$

$$\frac{d}{dr} = \frac{dt}{dr} \frac{d}{dt} = \frac{1}{\dot{r}} \frac{d}{dt}$$

$$\frac{d^2}{dr^2} = \frac{1}{\dot{r}^2} \frac{d^2}{dt^2} - \frac{\ddot{r}}{\dot{r}^3} \frac{d}{dt}$$

$$\frac{d^2}{dr^2} = \frac{1}{\dot{r}^2} \frac{d^2}{dt^2} - \frac{\ddot{r}}{\dot{r}^3} \frac{d}{dt}$$

$$\frac{d^2 \theta}{dr^2} = - \frac{1}{r^2}$$

$$\frac{\ddot{\theta}}{\dot{r}^2} = \left[- \left(\frac{d\theta}{dt} \right)^2 \cot \theta - \frac{1}{r^2} \cot \theta - \frac{2}{r} \left(\frac{d\theta}{dr} \right) + \frac{\cot \theta}{r^2} \left[1 + r^2 \left(\frac{d\theta}{dr} \right)^2 + r^2 \sin^2 \theta \left(\frac{d\varphi}{dr} \right)^2 \right] + \frac{2M}{r^2} \left(\frac{d\theta}{dr} \right) \right]$$